Note on the center of generalized quantum groups (Quantum groups and quantum topology)

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Note on the center of generalized quantum groups

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1 Introduction

Recently study of generalized quantum groups defined for wider class of bi-characters has been achieved marvelously. It can be said that the study was initiated by Andruskiewitsch and Schneider's suggestion [2] of classification program of pointed Hopf algebras. It should be mentioned that the Drinfeld-Jimbo's original quantum groups, the Lusztig small quantum groups at roots of unity, the quantum superalgebras of type A-G, the ones at roots of unity, $\mathbb{Z}/3\mathbb{Z}$-quatum groups (cf. [8]), and multi-parameter quantum groups are generalized quantum groups. Under the idea, Heckenberger achieved studies of Nichols algebras of diagonal type and their quantum doubles, including classification of those of finite type [3], [4], [5]. Under influence of the program, he and the author obtained a Matsumoto type theorem of the Weyl groupoids associated to finite type generalized root systems [6]. Algebras mentioned above admit generalized root systems. They also obtained a factorization formula of the Shapovalov determinants of finite type generalized quantum groups [7].

We consider that it is very important to formulate a Harish-Chandra theorem for generalized quantum groups. In this note, we make preliminary study of the Harish-Chandra maps of generalized quantum groups defined for symmetric bi-characters. For original results for quantum groups, see [1].

2 Multi-parameter generalized quantum groups

For x, $y\in\mathbb{Z}$, let $J_{x,y}:=\{z\in\mathbb{Z}|x\leq z\leq y\}$. Let $SJ_{x,y}$ be the set of finite sequences in $J_{x,y}$; we assume that $SJ_{x,y}$ has a 0-sequence ϕ. Namely $SJ_{x,y}=\cup_{r=0}^{+\infty}SJ_{x,y}^{(r)}$ (disjoint), where $SJ_{x,y}^{(0)}=\{\phi\}$, and if $r\in\mathbb{N}$, we mean $SJ_{x,y}^{(r)}=\{(i_1,i_2,\ldots,i_r)|i_{r'}\in J_{x,y}(r'\in J_{x,y})\}$. For $\bar{x}=(i_1,i_2,\ldots,i_r)\in SJ_{x,y}^{(r)}$, let $||\bar{x}||=(j_1,j_2,\ldots,j_r)\in SJ_{x,y}^{(r)}$ be such that $j_1\leq j_2\leq\ldots\leq j_r$ and $|\{k\in J_{1,r}|j_k=z\}|=|\{l\in J_{1,r}|i_l=z\}|$ for all $z\in J_{x,y}$; we also let $||\phi||:=\phi$.

Let $N\in\mathbb{N}$. Let $q\in\mathbb{C}\setminus\bar{\mathbb{Q}}$. Fix $q^{\frac{1}{2}}\in\mathbb{C}\setminus\bar{\mathbb{Q}}$ with $q=(q^{\frac{1}{2}})^2$. For $r\in\mathbb{Z}$, we write $q^{\frac{r}{2}}:=(q^{\frac{1}{2}})^r$. Let $\Pi''=\{\epsilon_i,{\epsilon'}_i|i\in J_{1,N}\}$ be a finite set with $2N=|\Pi''|$. Let $\mathbb{Z}\Pi''$ be a rank-$2N$ free $\mathbb{Z}$-module with the base Π''. Let $\sqrt{\chi}:\mathbb{Z}\Pi''\times\mathbb{Z}\Pi''\to\mathbb{C}^\times$ be a map such that

$$(2.1)\qquad \sqrt{\chi}(a,b+c)=\sqrt{\chi}(a,b)\sqrt{\chi}(a,c),\ \sqrt{\chi}(a+b,c)=\sqrt{\chi}(a,c)\sqrt{\chi}(b,c)$$

for all a, b, $c \in \mathbb{Z}\Pi''$, and

$$\sqrt{\chi}(\epsilon_i, \epsilon'_j) = \sqrt{\chi}(\epsilon'_i, \epsilon_j) = q^{\frac{\delta_{ij}}{2}}, \quad \sqrt{\chi}(\epsilon'_i, \epsilon'_j) = 1 \tag{2.2}$$

for all i, $j \in J_{1,N}$. Define a map $\chi : \mathbb{Z}\Pi'' \times \mathbb{Z}\Pi'' \to \mathbb{C}^\times$ by

$$\chi(a, b) := \sqrt{\chi}(a, b)^2. \tag{2.3}$$

Let $\Pi' := \{\epsilon_j | j \in J_{1,N}\}$, so $\Pi' \subset \Pi''$. Let $\ell \in J_{1,N}$. Let $\Pi = \{\alpha_i | i \in J_{1,\ell}\}$ be a subset of $\mathbb{Z}\Pi'$ such that $\mathbb{Z}\Pi$ is a rank-ℓ submodule of $\mathbb{Z}\Pi'$.

Let $\tilde{B}^+ = \tilde{B}^+(\chi)$ be the unital $\mathbb{C}$-algebra defined with generators

$$L_a \, (a \in \mathbb{Z}\Pi''), \;\; E_i \, (i \in J_{1,\ell}) \tag{2.4}$$

and relations

$$L_0 = 1, \;\; L_a L_b = L_{a+b}, \;\; L_a E_i = \chi(a, \alpha_i) E_i L_a. \tag{2.5}$$

For $\phi \in SJ_{1,\ell}{}^{(0)}$, let $E_\phi := 1 \in \tilde{B}^+$, and for $\bar{x} := (i_1, \ldots, i_r) \in SJ_{1,\ell}{}^{(r)}$ with $r \in \mathbb{N}$, let $\bar{E}_{\bar{x}} := E_{i_1} \cdots E_{i_r}$. Then using a standard argument, we see

Lemma 1. *As a $\mathbb{C}$-linear space, $\tilde{B}^+$ has a $\mathbb{C}$-basis*

$$\{\bar{E}_{\bar{x}} L_a \,|\, \bar{x} \in SJ_{1,\ell}, \; a \in \mathbb{Z}\Pi''\}. \tag{2.6}$$

The $\mathbb{C}$-algebra $\tilde{B}^+$ can be regarded as a Hopf algebra $(\tilde{B}^+, \Delta, S, \varepsilon)$ with $\Delta(L_a) = L_a \otimes L_a$, $S(L_a) = L_{-a}$, $\varepsilon(L_a) = 1$, $\Delta(E_i) = E_i \otimes 1 + L_{\alpha_i} \otimes E_i$, $S(E_i) = -L_{-\alpha_i} E_i$, $\varepsilon(E_i) = 0$.

Let $(\tilde{B}^+)^*$ be the dual linear space of $\tilde{B}^+$. We regard $(\tilde{B}^+)^*$ as a $\tilde{B}^+$-module by $X.f(Y) = f(YX)$ for all $f \in (\tilde{B}^+)^*$, and all X, $Y \in \tilde{B}^+$. Let

$$(\tilde{B}^+)^\circ := \{f \in (\tilde{B}^+)^* \mid \dim \tilde{B}^+.f < +\infty\}. \tag{2.7}$$

Let $f \in (\tilde{B}^+)^\circ \setminus \{0\}$. Let $r := \dim \tilde{B}^+.f$ and let $\{f_i | i \in J_{1,r}\}$ be a $\mathbb{C}$-basis of $\tilde{B}^+.f$. Assume that $f_1 = f$. Define $\rho_{ij} \in (\tilde{B}^+)^*$ $(i, j \in J_{1,r})$ by $X.f_j = \sum_{i \in J_{1,r}} \rho_{ij}(X) f_i$. Then $\rho_{ij}(XY) = \sum_{k \in J_{1,r}} \rho_{ik}(X)\rho_{kj}(Y)$, so $\rho_{ij} \in (\tilde{B}^+)^\circ$. We have $f = \sum_{i \in J_{1,r}} f_i(1)\rho_{ij}$. We regard $(\tilde{B}^+)^\circ$ as a unital $\mathbb{C}$-algebra with the unit ε, by the multiplication defined by $fg(X) := \sum_k f(X_k^{(1)}) g(X_k^{(2)})$ for all f, $g \in (\tilde{B}^+)^\circ$, and all $X \in \tilde{B}^+$ with $\Delta(X) = \sum_k X_k^{(1)} \otimes X_k^{(2)}$; we note that $fg \in (\tilde{B}^+)^\circ$ since $X.(fg) = \sum_k (X_k^{(1)}.f)(X_k^{(2)}.g)$. We regard $(\tilde{B}^+)^\circ$, $(\tilde{B}^+)^\circ \otimes (\tilde{B}^+)^\circ$, $(\tilde{B}^+)^* \otimes (\tilde{B}^+)^*$ as subspaces of $(\tilde{B}^+)^*$, $(\tilde{B}^+)^* \otimes (\tilde{B}^+)^*$, $(\tilde{B}^+ \otimes \tilde{B}^+)^*$ respectively in a natural way. Define the linear maps $\Delta^\circ : (\tilde{B}^+)^\circ \to (\tilde{B}^+)^\circ \otimes (\tilde{B}^+)^\circ (\subset (\tilde{B}^+)^* \otimes (\tilde{B}^+)^* \subset (\tilde{B}^+ \otimes \tilde{B}^+)^*)$, $S^\circ : (\tilde{B}^+)^\circ \to (\tilde{B}^+)^\circ (\subset (\tilde{B}^+)^*)$ and $\varepsilon^\circ : (\tilde{B}^+)^\circ \to \mathbb{C}$ by $\Delta^\circ(f)(X \otimes Y) = f(XY)$, $S^\circ(f)(X) = f(S(X))$ and $\varepsilon^\circ(f) = f(1)$ respectively, where we note that $\Delta^\circ(\rho_{ij}) = \sum_{k\in J_{1,r}} \rho_{ik}\otimes\rho_{kj}$, $X.(S^\circ(\rho_{ij})) = \sum_{k\in J_{1,r}} \rho_{ik}(S(X))S^\circ(\rho_{kj})$, and $\sum_{k\in J_{1,r}} \varepsilon^\circ(\rho_{ik})\rho_{kj} = \sum_{k\in J_{1,r}} \varepsilon^\circ(\rho_{kj})\rho_{ik} = \rho_{ij}$ hold for the above ρ_{ij}. Then

$$(\tilde{B}^+)^\circ = ((\tilde{B}^+)^\circ, \Delta^\circ, S^\circ, \varepsilon^\circ). \tag{2.8}$$

can be regarded as a Hopf algebra. For the above χ, define the map $\chi^\vee : \mathbb{Z}\Pi'' \times \mathbb{Z}\Pi'' \to \mathbb{C}^\times$ by $\chi^\vee(a,b) := \chi(b,a)$, and let $(\tilde{B}^+)^\vee := \tilde{B}^+(\chi^\vee)$. We denote the elements L_a, E_i and $\bar{E}_{\bar{x}}$ of $(\tilde{B}^+)^\vee$ by $L_a^\vee$, $E_i^\vee$ and $\bar{E}_{\bar{x}}^\vee$ respectively. By a natural argument, we see

Lemma 2. *There exists a unique Hopf algebra homomorphism $\varphi : (\tilde{B}^+)^\vee \to (\tilde{B}^+)^\circ$ such that $\varphi(L_a^\vee)(\bar{E}_{\bar{x}}L_b) = \delta_{\bar{x},\phi}\chi(b,a)$ and $\varphi(E_i^\vee)(\bar{E}_{\bar{x}}L_b) = \delta_{\bar{x},(i)}$.*

Define the bi-linear map

$$(\,,\,) : \tilde{B}^+ \times (\tilde{B}^+)^\vee \to \mathbb{C} \tag{2.9}$$

by $(X, X^\vee) := \varphi(X^\vee)(X)$. We denote the maps Δ, S and ε for $(\tilde{B}^+)^\vee$ by $\Delta^\vee$, $S^\vee$ and $\varepsilon^\vee$ respectively. As a bi-linear map, $(\,,\,)$ is characterized by

$$\begin{aligned} &(L_a, L_b^\vee) = \chi(a,b),\ (E_i, E_j^\vee) = \delta_{ij}, (L_a, E_j^\vee) = (E_i, L_b^\vee) = 0,\\ &(XY, X^\vee) = \textstyle\sum_r (X, (X^\vee)_r^{(1)})(Y, (X^\vee)_r^{(2)}),\\ &(X, X^\vee Y^\vee) = \textstyle\sum_k (X_k^{(1)}, X^\vee)(X_k^{(2)}, Y^\vee), \end{aligned} \tag{2.10}$$

for all $a, b \in \mathbb{Z}\Pi''$, all $i, j \in J_{1,\ell}$, all $X, Y \in \tilde{B}^+$ with $\Delta(X) = \sum_k X_k^{(1)} \otimes X_k^{(2)}$, and all $X^\vee, Y^\vee \in \tilde{B}^+$ with $\Delta^\vee(X^\vee) = \sum_k (X^\vee)_k^{(1)} \otimes (X^\vee)_k^{(2)}$. We also have

$$(\bar{E}_{\bar{x}}L_a, \bar{E}_{\bar{y}}^\vee L_b^\vee) = \delta_{||\bar{x}||,||\bar{y}||} \cdot \chi(a,b)(\bar{E}_{\bar{x}}, \bar{E}_{\bar{y}}^\vee) \tag{2.11}$$

for all $\bar{x}$, $\bar{y} \in SJ_{1,\ell}$ and all a, $b \in \mathbb{Z}\Pi''$. We also have

$$(S(X), X^\vee) = (X, S^\vee(X^\vee)),\ (1, X^\vee) = \varepsilon^\vee(X^\vee),\ (X, 1) = \varepsilon(X) \tag{2.12}$$

for all $X \in \tilde{B}^+$ and all $X^\vee \in (\tilde{B}^+)^\vee$. Let $\tilde{C}^+ := \{X \in \tilde{B}^+ | (X, (\tilde{B}^+)^\vee) = \{0\}\}$ and $(\tilde{C}^+)^\vee := \{X^\vee \in (\tilde{B}^+)^\vee | (\tilde{B}^+, X^\vee) = \{0\}\}$. Let B^+ and $(B^+)^\vee$ denote the quotient Hopf algebras $\tilde{B}^+/\tilde{C}^+$ and $(\tilde{B}^+)^\vee/(\tilde{C}^+)^\vee$ respectively. By abuse of notation, we shall use the same symbols for objects L_a, E_i, $L_a^\vee$, $E_i^\vee$, $(\,,\,)$ etc. and the objects defined as those modulo $\tilde{C}^+$ or/and $(\tilde{C}^+)^\vee$.

By a cerebrate argument due to Drinfeld, we have a Hopf algebra $D = D(\sqrt{\chi}) = (D = D(\sqrt{\chi}), \Delta = \Delta_D, S = S_D, \varepsilon = \varepsilon_D)$ such that

(1) As a $\mathbb{C}$-linear space, $D = B^+ \otimes (B^+)^\vee = B^+(\chi) \otimes B^+(\chi^\vee)$. By abuse of notation, for $X \in B^+$ and $X^\vee \in (B^+)^\vee$, we denote the elements $X \otimes 1$ and $1 \otimes X^\vee$ of D by X and $X^\vee$ respectively. The linear map $B^+ \to D$, $X \mapsto X$, is a Hopf algebra homomorphism. The linear map $(B^+)^\vee \to D$, $X^\vee \mapsto X^\vee$, is a $\mathbb{C}$-algebra

homomorphism. For $X^\vee \in (B^+)^\vee$ with $\Delta^\vee(X^\vee) = \sum_r (X^\vee)^{(1)}_r \otimes (X^\vee)^{(2)}_r$, we have $\Delta_D(X^\vee) = \sum_r (X^\vee)^{(2)}_r \otimes (X^\vee)^{(1)}_r$, $S_D(X^\vee) = (S^\vee)^{-1}(X^\vee)$, and $\varepsilon_D(X^\vee) = \varepsilon^\vee(X^\vee)$.

(2) As for the multiplication of D, for $X \in B^+$ and $X^\vee \in (B^+)^\vee$, with $((1\otimes\Delta)\circ\Delta)(X) = \sum_r X^{(1)}_r \otimes X^{(2)}_r \otimes X^{(3)}_r$ and $((1\otimes\Delta^\vee)\circ\Delta^\vee)(X^\vee) = \sum_k (X^\vee)^{(1)}_k \otimes (X^\vee)^{(2)}_k \otimes (X^\vee)^{(3)}_k$, we have

$$X^\vee \cdot X = \sum_{r,k} (S^{-1}(X^{(1)}_r), (X^\vee)^{(3)}_k)(X^{(3)}_r, (X^\vee)^{(1)}_k) X^{(2)}_r \cdot (X^\vee)^{(2)}_k. \tag{2.13}$$

For X and $X^\vee$ in the above (2), we also have

$$X \cdot X^\vee = \sum_{r,k} (S^{-1}(X^{(3)}_r), (X^\vee)^{(1)}_k)(X^{(1)}_r, (X^\vee)^{(3)}_k)(X^\vee)^{(2)}_k \cdot X^{(2)}_r. \tag{2.14}$$

Further we have

$$\begin{aligned} &L_a L^\vee_b = L^\vee_b L_a,\ L_a E^\vee_j = \chi(-a, \alpha_j) E^\vee_j L_a,\ L^\vee_b E_i = \chi(\alpha_i, -b) E_i L^\vee_b, \\ &E_i E^\vee_j - E^\vee_j E_i = \delta_{ij}(-L_{\alpha_i} + L^\vee_{-\alpha_i}). \end{aligned} \tag{2.15}$$

Note that $L_0 = L^\vee_0 = 1$ holds in D. Let $\mathbb{Z}_{\geq 0}\Pi = \{\sum_{i\in J_{1,\ell}} n_i\alpha_i \in \mathbb{Z}\Pi | n_i \in \mathbb{Z},\ n_i \geq 0\}$. For $\beta \in \mathbb{Z}\Pi$, define subspaces U^+_β and $U^-_{-\beta}$ of D in the following way. If $\beta \in \mathbb{Z}\Pi \setminus \mathbb{Z}_{\geq 0}\Pi$, let $U^+_\beta := U^-_{-\beta} := \{0\}$. Let $U^+_0 := U^-_0 := \mathbb{C}L_0$. If $\beta \in \mathbb{Z}_{\geq 0}\Pi$, let $U^+_\beta := \sum_{i\in J_{1,\ell}} E_i U^+_{\beta-\alpha_i}$, and $U^-_{-\beta} := \sum_{i\in J_{1,\ell}} E^\vee_i U^-_{-\beta+\alpha_i}$. Let $U^+ := \sum_{\beta\in\mathbb{Z}_{\geq 0}\Pi} U^+_\beta$ and $U^- := \sum_{\beta\in\mathbb{Z}_{\geq 0}\Pi} U^-_{-\beta}$. Let $D^0 := \sum_{\gamma,\theta\in\mathbb{Z}\Pi''} \mathbb{C}L_\gamma L^\vee_\theta$. Then $D = \mathrm{Span}_{\mathbb{C}}(U^+D^0U^-) = \mathrm{Span}_{\mathbb{C}}(U^-D^0U^+)$. Further, by (2.11), we have

Lemma 3. (1) *For any γ, $\theta \in \mathbb{Z}\Pi''$, $L_\gamma L^\vee_\theta \neq 0$ holds in D. In particular,* $\dim U^+_0 = \dim U^-_0 = 1$

(2) $\dim U^+_{\alpha_i} = \dim U^-_{-\alpha_i} = 1$ *holds for any $i \in J_{1,\ell}$.*

(3) $U^+ = \oplus_{\beta\in\mathbb{Z}_{\geq 0}\Pi} U^+_\beta$, $U^- = \oplus_{\beta\in\mathbb{Z}_{\geq 0}\Pi} U^-_{-\beta}$, *and* $D^0 = \oplus_{\gamma,\theta\in\mathbb{Z}\Pi''}\mathbb{C}L_\gamma L^\vee_\theta$ *hold as $\mathbb{C}$-linear spaces.*

(4) *The linear maps $U^+ \otimes D^0 \otimes U^- \to D$, $X \otimes L_\gamma L^\vee_\theta \otimes X^\vee \mapsto X L_\gamma L^\vee_\theta X^\vee$, and $U^- \otimes D^0 \otimes U^+ \to D$, $X^\vee \otimes L_\gamma L^\vee_\theta \otimes X \mapsto X^\vee L_\gamma L^\vee_\theta X$, are bijective.*

3 Rosso form

From now on, except for Section 8, we assume that $\sqrt{\chi}$ is symmetric, that is,

(3.1) we assume that $\sqrt{\chi}(a, b) = \sqrt{\chi}(b, a)$ for all a, $b \in \mathbb{Z}\Pi''$.

Define the subgroup T of $\mathbb{Z}\Pi'$ by

$$T := \{\omega \in \mathbb{Z}\Pi' | \sqrt{\chi}(\omega, \mathbb{Z}\Pi') = \{1\}\}. \tag{3.2}$$

Let D' be the subalgebra of D generated by E_i, $E_i^\vee$ ($i \in J_{1,\ell}$) and L_θ, $L_\theta^\vee$ ($\theta \in \mathbb{Z}\Pi'$). Then D' is a Hopf subalgebra of D. Let G be the ideal of D' (as a $\mathbb{C}$-algebra) generated by $L_\theta L_\theta^\vee - 1$ ($\theta \in \mathbb{Z}\Pi'$) and $L_\omega - 1$ ($\omega \in T$). Let $U = U(\sqrt{\chi}) := D'/G$ (as a $\mathbb{C}$-algebra). Then U can be regarded as a quotient Hopf algebra of D'. Let

$$\overline{\mathbb{Z}\Pi'} := \mathbb{Z}\Pi'/T. \tag{3.3}$$

For $\lambda \in \mathbb{Z}\Pi'$, let $\bar{\lambda} := \lambda + T \in \overline{\mathbb{Z}\Pi'}$, and let $L_{\bar{\lambda}} := L_\lambda + G \in U$. For any $\eta \in \overline{\mathbb{Z}\Pi'}$, $L_\eta \neq 0$ holds in U. Let $U^0 := \sum_{\eta \in \overline{\mathbb{Z}\Pi'}} L_\eta$. Then $U^0 = \oplus_{\eta \in \overline{\mathbb{Z}\Pi'}} L_\eta$ holds. The U^+ and U^- in the previous section can be regarded as subalgeras of U. Further, the linear maps $U^+ \otimes U^0 \otimes U^- \to D$, $X \otimes L_\eta \otimes X^\vee \mapsto X L_\eta X^\vee$, and $U^- \otimes U^0 \otimes U^+ \to D$, $X^\vee \otimes L_\eta \otimes X \mapsto X^\vee L_\eta X$, are bijective. We have a $\mathbb{C}$-algebra automorphism Ω of U such that $\Omega(E_i) = E_i^\vee$, $\Omega(E_i^\vee) = E_i$, and $\Omega(L_\eta) = L_{-\eta}$. Then $\Omega^2 = 1$. Let $U^{\geq 0} := \mathrm{Span}(U^+U^0)$. Define the bi-linear form $(\,,\,) : U^{\geq 0} \times U^{\geq 0} \to \mathbb{C}$ by $(XL_{\bar{\lambda}}, \tilde{X}L_{\bar{\mu}}) := (XL_\lambda, \Omega(\tilde{X})L^\vee_{-\mu})$ for all X, $\tilde{X} \in U^+$, and all λ, $\mu \in \mathbb{Z}\Pi'$. Then $(\,,\,)$ is symmetric. Define the non-degenerate bi-linear form

$$\langle\,,\,\rangle : U \times U \to \mathbb{C} \tag{3.4}$$

by

$$\langle XL_{\bar{\lambda}}S(Y), \tilde{Y}L_{\bar{\mu}}S(\tilde{X})\rangle := \sqrt{\chi}(-\lambda,\mu)(X,\Omega(\tilde{Y}))(\tilde{X},\Omega(Y)) \tag{3.5}$$

for all X, $\tilde{X} \in U^+$, all Y, $\tilde{Y} \in U^-$, and all λ, $\mu \in \mathbb{Z}\Pi'$.

Define the left action ad and the right action $\widetilde{\mathrm{ad}}$ of U on U by

$$\mathrm{ad}(u)\cdot v := \sum_r u_r^{(1)} v S(u_r^{(2)}) \quad \text{and} \quad v \cdot \widetilde{\mathrm{ad}}(u) := \sum_r S(u_r^{(1)}) v u_r^{(2)} \tag{3.6}$$

respectively for all u, $v \in U$ with $\Delta(u) = \sum_r u_r^{(1)} \otimes u_r^{(2)}$.

Theorem 4. *We have*

$$\langle \mathrm{ad}(u)\cdot v_1, v_2\rangle = \langle v_1, v_2 \cdot \widetilde{\mathrm{ad}}(u)\rangle \tag{3.7}$$

for all u, v_1, $v_2 \in U$.

Proof. We may assume that

$$v_1 = XL_{\bar{\lambda}}S(Y) \quad \text{and} \quad v_2 = \tilde{Y}L_{\bar{\mu}}S(\tilde{X}) \tag{3.8}$$

with λ, $\mu \in \mathbb{Z}\Pi'$, $X \in U_\theta^+$, $Y \in U_{-\gamma}^-$, $\tilde{X} \in U_\omega^+$, $\tilde{Y} \in U_{-\delta}^-$, and θ, γ, ω, $\delta \in \mathbb{Z}_{\geq 0}\Pi$.

Case-1. Assume $u = L_{\bar{\nu}}$ with $\nu \in \mathbb{Z}\Pi'$. Then we have

$$\begin{aligned} \langle \mathrm{ad}(u)\cdot v_1, v_2\rangle &= \chi(\nu,\theta-\gamma)\langle v_1, v_2\rangle \\ &= \chi(\nu,\theta-\gamma)\delta_{\theta,\delta}\delta_{\gamma,\omega}\langle v_1,v_2\rangle = \chi(-\nu,\omega-\delta)\langle v_1,v_2\rangle \\ &= \langle v_1, v_2\cdot\widetilde{\mathrm{ad}}(u)\rangle, \end{aligned}$$

as desired.

Case-2. Assume $u \in U_\beta^+$ with $\beta \in \mathbb{Z}_{\geq 0}\Pi$. We write:

$$\Delta(u) = \sum_{r'} u_{r'}^{(1)} \otimes u_{r'}^{(2)}, \tag{3.9}$$

$$\begin{aligned}
&((1 \otimes 1 \otimes \Delta) \circ (1 \otimes \Delta) \circ \Delta)(u) \\
&= \sum_{r''} u_{r''}^{(1)} \otimes u_{r''}^{(2)} \otimes u_{r''}^{(3)} \otimes u_{r''}^{(4)} \\
&= \sum_{\vec{\beta},r} u_{1,r}^{(\beta_1)} L_{\overline{\beta_2+\beta_3+\beta_4}} \otimes u_{2,r}^{(\beta_2)} L_{\overline{\beta_3+\beta_4}} \otimes u_{3,r}^{(\beta_3)} L_{\overline{\beta_4}} \otimes u_{4,r}^{(\beta_4)},
\end{aligned}$$

where $\vec{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4) \in (\mathbb{Z}_{\geq 0}\Pi)^4$ with $\beta_1 + \beta_2 + \beta_3 + \beta_4 = \beta$, and $u_{x,r}^{(\beta_x)} \in U_{\beta_x}^+$. We also write:

$$\begin{aligned}
&((1 \otimes \Delta) \circ \Delta)(Y) \\
&= \sum_{s'} Y_{s'}^{(1)} \otimes Y_{s'}^{(2)} \otimes Y_{s'}^{(3)} \\
&= \sum_{\vec{\gamma},s} Y_{1,s}^{(\gamma_1)} \otimes Y_{2,s}^{(\gamma_2)} L_{\overline{-\gamma_1}} \otimes Y_{3,s}^{(\gamma_3)} L_{\overline{-\gamma_1-\gamma_2}},
\end{aligned} \tag{3.10}$$

where $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{Z}_{\geq 0}\Pi)^3$ with $\gamma_1 + \gamma_2 + \gamma_3 = \gamma$, and $Y_{y,s}^{(\gamma_y)} \in U_{-\gamma_y}^-$. Then

we have

$$\begin{aligned}
\mathrm{ad}(u)\cdot v_1 &= \sum_{r'} u_{r'}^{(1)} v_1 S(u_{r'}^{(2)}) \\
&= \sum_{r'} u_{r'}^{(1)} X L_{\bar\lambda} S(Y) S(u_{r'}^{(2)}) = \sum_{r'} u_{r'}^{(1)} X L_{\bar\lambda} S(u_{r'}^{(2)} Y) \\
&= \sum_{r'',s'} u_{r''}^{(1)} X L_{\bar\lambda} S\left((u_{r''}^{(2)}, \Omega(Y_{s'}^{(1)}))(S^{-1}(u_{r''}^{(4)}), \Omega(Y_{s'}^{(3)})) Y_{s'}^{(2)} u_{r''}^{(3)} \right) \\
&= \sum_{\vec\beta,r,\vec\gamma,s} (u_{2,r}^{(\beta_2)} L_{\overline{\beta_3+\beta_4}}, \Omega(Y_{1,s}^{(\gamma_1)}))(S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)} L_{\overline{-\gamma_1-\gamma_2}})) \\
&\qquad \cdot u_{1,r}^{(\beta_1)} L_{\overline{\beta_2+\beta_3+\beta_4}} X L_{\bar\lambda} S(Y_{2,s}^{(\gamma_2)} L_{\overline{-\gamma_1}} u_{3,r}^{(\beta_3)} L_{\overline{\beta_4}}) \\
&= \sum_{\vec\beta,r,\vec\gamma,s} \chi(\beta_2+\beta_3+\beta_4,\theta)\chi(-\beta_4,\beta_3) \\
&\qquad \cdot (u_{2,r}^{(\beta_2)} L_{\overline{\beta_3+\beta_4}}, \Omega(Y_{1,s}^{(\gamma_1)}))(S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)} L_{\overline{-\gamma_1-\gamma_2}})) \\
&\qquad \cdot u_{1,r}^{(\beta_1)} X L_{\overline{\beta_2+\beta_3+\beta_4+\lambda}} S(Y_{2,s}^{(\gamma_2)} L_{\overline{-\gamma_1+\beta_4}} u_{3,r}^{(\beta_3)}) \\
&= \sum_{\vec\beta,r,\vec\gamma,s} \chi(\beta_2+\beta_3+\beta_4,\theta)\chi(-\beta_4,\beta_3)\chi(\beta_2+\beta_3+\beta_4+\lambda,\beta_3) \\
&\qquad \cdot (u_{2,r}^{(\beta_2)} L_{\overline{\beta_3+\beta_4}}, \Omega(Y_{1,s}^{(\gamma_1)}))(S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)} L_{\overline{-\gamma_1-\gamma_2}})) \\
&\qquad \cdot (u_{1,r}^{(\beta_1)} X S(u_{3,r}^{(\beta_3)}) L_{\overline{\beta_3}}) L_{\overline{\beta_2+\lambda+\gamma_1}} S(Y_{2,s}^{(\gamma_2)}) \\
&= \sum_{\vec\beta,r,\vec\gamma,s} \chi(\beta_2+\beta_3+\beta_4,\theta)\chi(\beta_2+\beta_3+\lambda,\beta_3) \\
&\qquad \cdot (u_{2,r}^{(\beta_2)} L_{\overline{\beta_3+\beta_4}}, \Omega(Y_{1,s}^{(\gamma_1)}))(S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)} L_{\overline{-\gamma_1-\gamma_2}})) \\
&\qquad \cdot (u_{1,r}^{(\beta_1)} X S(u_{3,r}^{(\beta_3)}) L_{\overline{\beta_3}}) L_{\overline{\beta_2+\lambda+\gamma_1}} S(Y_{2,s}^{(\gamma_2)}).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\langle \mathrm{ad}(u)\cdot v_1, v_2\rangle \\
&= \sum_{\vec{\beta},r,\vec{\gamma},s} \chi(\beta_2+\beta_3+\beta_4,\theta)\chi(\beta_2+\beta_3+\lambda,\beta_3)\sqrt{\chi}(-(\beta_2+\lambda+\gamma_1),\mu) \\
&\qquad \cdot(u_{2,r}^{(\beta_2)}L_{\overline{\beta_3+\beta_4}}, \Omega(Y_{1,s}^{(\gamma_1)}))(S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)}L_{\overline{-\gamma_1-\gamma_2}})) \\
&\qquad \cdot(u_{1,r}^{(\beta_1)}XS(u_{3,r}^{(\beta_3)})L_{\overline{\beta_3}}, \Omega(\tilde{Y}))(\tilde{X}, \Omega(Y_{2,s}^{(\gamma_2)})) \\
&= \sum_{\vec{\beta},r,\vec{\gamma},s} \chi(\beta_2+\beta_3+\beta_4,\theta)\chi(\beta_2+\beta_3+\lambda,\beta_3)\sqrt{\chi}(-(\beta_2+\lambda+\gamma_1),\mu) \\
&\qquad \cdot\delta_{\beta_2,\gamma_1}\delta_{\beta_4,\gamma_3}\delta_{\beta_1+\theta+\beta_3,\delta}\delta_{\omega,\gamma_2} \\
&\qquad \cdot(u_{2,r}^{(\beta_2)}, \Omega(Y_{1,s}^{(\gamma_1)}))(S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)}L_{\overline{-\gamma_1-\gamma_2}})) \\
&\qquad \cdot(u_{1,r}^{(\beta_1)}XS(u_{3,r}^{(\beta_3)})L_{\overline{\beta_3}}, \Omega(\tilde{Y}))(\tilde{X}, \Omega(Y_{2,s}^{(\gamma_2)}L_{\overline{-\gamma_1}})) \\
&= \sum_{\vec{\beta},r} \chi(\beta_2+\beta_3+\beta_4,\theta)\chi(\beta_2+\beta_3+\lambda,\beta_3)\chi(-\beta_2,\mu)\sqrt{\chi}(-\lambda,\mu) \\
&\qquad \cdot(S^{-1}(u_{4,r}^{(\beta_4)})\tilde{X}u_{2,r}^{(\beta_2)}, \Omega(Y))(u_{1,r}^{(\beta_1)}XS(u_{3,r}^{(\beta_3)})L_{\overline{\beta_3}}, \Omega(\tilde{Y})).
\end{aligned}$$

We write:

$$\begin{aligned}
&((1\otimes\Delta)\circ\Delta)(\tilde{Y}) \\
&= \sum_{t'} \tilde{Y}_{t'}^{(1)} \otimes \tilde{Y}_{t'}^{(2)} \otimes \tilde{Y}_{t'}^{(3)} \\
&= \sum_{\vec{\delta},t} \tilde{Y}_{1,t}^{(\delta_1)} \otimes \tilde{Y}_{2,t}^{(\delta_2)}L_{\overline{-\delta_1}} \otimes \tilde{Y}_{3,t}^{(\delta_3)}L_{\overline{-\delta_1-\delta_2}},
\end{aligned}$$

where $\vec{\delta}=(\delta_1,\delta_2,\delta_3)\in(\mathbb{Z}_{\geq 0}\Pi)^3$ with $\delta_1+\delta_2+\delta_3=\delta$, and $\tilde{Y}_{y,s}^{(\delta_z)}\in U_{-\delta_z}^-$. Then

we have

$$
\begin{aligned}
v_1 \cdot \widetilde{\mathrm{ad}}(u) &= \sum_{r'} S(u_{r'}^{(1)}) v_2 u_{r'}^{(2)} \\
&= \sum_{r'} S(u_{r'}^{(1)}) \tilde{Y} L_{\bar{\mu}} S(\tilde{X}) u_{r'}^{(2)} \\
&= \sum_{r'',t'} (S(u_{r''}^{(3)}), \Omega(\tilde{Y}_{t'}^{(1)}))(S^{-1}(S(u_{r''}^{(1)})), \Omega(\tilde{Y}_{t'}^{(3)})) \tilde{Y}_{t'}^{(2)} S(u_{r''}^{(2)}) L_{\bar{\mu}} S(\tilde{X}) u_{r''}^{(4)} \\
&= \sum_{\vec{\beta},r,\vec{\delta},t} (S(u_{3,r}^{(\beta_3)} L_{\overline{\beta_4}}), \Omega(\tilde{Y}_{1,t}^{(\delta_1)}))(u_{1,r}^{(\beta_1)} L_{\overline{\beta_2+\beta_3+\beta_4}}, \Omega(\tilde{Y}_{3,t}^{(\delta_3)} L_{\overline{-\delta_1-\delta_2}})) \\
&\qquad \cdot \tilde{Y}_{2,t}^{(\delta_2)} L_{\overline{-\delta_1}} S(u_{2,r}^{(\beta_2)} L_{\overline{\beta_3+\beta_4}}) L_{\bar{\mu}} S(\tilde{X}) u_{4,r}^{(\beta_4)} \\
&= \sum_{\vec{\beta},r,\vec{\delta},t} (S(u_{3,r}^{(\beta_3)} L_{\overline{\beta_4}}), \Omega(\tilde{Y}_{1,t}^{(\delta_1)}))(u_{1,r}^{(\beta_1)} L_{\overline{\beta_2+\beta_3+\beta_4}}, \Omega(\tilde{Y}_{3,t}^{(\delta_3)} L_{\overline{-\delta_1-\delta_2}})) \\
&\qquad \cdot \tilde{Y}_{2,t}^{(\delta_2)} L_{\overline{-\delta_1}} S(u_{2,r}^{(\beta_2)} L_{\overline{\beta_3+\beta_4}}) L_{\bar{\mu}} S(\tilde{X}) L_{\overline{\beta_4}} S(S^{-1}(u_{4,r}^{(\beta_4)}) L_{\overline{\beta_4}}) \\
&= \sum_{\vec{\beta},r,\vec{\delta},t} \chi(-\mu, \beta_2) \chi(-\beta_4, \beta_2 + \omega) \\
&\qquad \cdot (S(u_{3,r}^{(\beta_3)} L_{\overline{\beta_4}}), \Omega(\tilde{Y}_{1,t}^{(\delta_1)}))(u_{1,r}^{(\beta_1)} L_{\overline{\beta_2+\beta_3+\beta_4}}, \Omega(\tilde{Y}_{3,t}^{(\delta_3)} L_{\overline{-\delta_1-\delta_2}})) \\
&\qquad \cdot \tilde{Y}_{2,t}^{(\delta_2)} L_{\overline{-\delta_1-\beta_3+\mu}} S(S^{-1}(u_{4,r}^{(\beta_4)}) L_{\overline{\beta_4}} \tilde{X} u_{2,r}^{(\beta_2)}).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\langle v_1, v_2 \cdot \widetilde{\mathrm{ad}}(u)\rangle \\
&= \sum_{\vec{\beta},r,\vec{\delta},t} \chi(-\mu,\beta_2)\chi(-\beta_4,\beta_2+\omega)\sqrt{\chi}(-\lambda,-\delta_1-\beta_3+\mu) \\
&\qquad \cdot(S(u_{3,r}^{(\beta_3)}L_{\overline{\beta_4}}),\Omega(\tilde{Y}_{1,t}^{(\delta_1)}))(u_{1,r}^{(\beta_1)}L_{\overline{\beta_2+\beta_3+\beta_4}},\Omega(\tilde{Y}_{3,t}^{(\delta_3)}L_{\overline{-\delta_1-\delta_2}})) \\
&\qquad \cdot(X,\Omega(\tilde{Y}_{2,t}^{(\delta_2)}))(S^{-1}(u_{4,r}^{(\beta_4)})L_{\overline{\beta_4}}\tilde{X}u_{2,r}^{(\beta_2)},\Omega(Y)) \\
&= \sum_{\vec{\beta},r,\vec{\delta},t} \chi(-\mu,\beta_2)\chi(-\beta_4,\beta_2+\omega)\sqrt{\chi}(-\lambda,-\delta_1-\beta_3+\mu) \\
&\qquad \cdot\delta_{\beta_3,\delta_1}\delta_{\beta_1,\delta_3}\delta_{\theta,\delta_2}\delta_{\beta_4+\omega+\beta_2,\gamma} \\
&\qquad \cdot(S(u_{3,r}^{(\beta_3)}L_{\overline{\beta_4}}),\Omega(\tilde{Y}_{1,t}^{(\delta_1)}))(u_{1,r}^{(\beta_1)}L_{\overline{\beta_2+\beta_3+\beta_4}},\Omega(\tilde{Y}_{3,t}^{(\delta_3)}L_{\overline{-\delta_1-\delta_2}})) \\
&\qquad \cdot(X,\Omega(\tilde{Y}_{2,t}^{(\delta_2)}L_{\overline{-\delta_1}}))(S^{-1}(u_{4,r}^{(\beta_4)})L_{\overline{\beta_4}}\tilde{X}u_{2,r}^{(\beta_2)},\Omega(Y)) \\
&= \sum_{\vec{\beta},r} \chi(-\mu,\beta_2)\chi(-\beta_4,\beta_2+\omega)\chi(\lambda,\beta_3)\sqrt{\chi}(-\lambda,\mu) \\
&\qquad \cdot(u_{1,r}^{(\beta_1)}L_{\overline{\beta_2+\beta_3+\beta_4}}XS(u_{3,r}^{(\beta_3)}L_{\overline{\beta_4}}),\Omega(\tilde{Y}))(S^{-1}(u_{4,r}^{(\beta_4)})L_{\overline{\beta_4}}\tilde{X}u_{2,r}^{(\beta_2)},\Omega(Y)) \\
&= \sum_{\vec{\beta},r} \chi(-\mu,\beta_2)\chi(-\beta_4,\beta_2+\omega)\chi(\lambda,\beta_3)\sqrt{\chi}(-\lambda,\mu) \\
&\qquad \cdot\chi(\beta_4,\theta)\chi(\beta_2+\beta_3,\theta+\beta_3)\chi(\beta_4,\omega+\beta_2) \\
&\qquad \cdot(u_{1,r}^{(\beta_1)}XS(u_{3,r}^{(\beta_3)}),\Omega(\tilde{Y}))(S^{-1}(u_{4,r}^{(\beta_4)})\tilde{X}u_{2,r}^{(\beta_2)},\Omega(Y)) \\
&= \langle \mathrm{ad}(u)\cdot v_1, v_2\rangle,
\end{aligned}
$$

as desired.

Case-3. Assume $u = E_i^\vee$ with $i \in J_{1,\ell}$. Note that $\Omega S \Omega = S^{-1}$. Then

$$
\begin{aligned}
&\langle \Omega(v_2), \Omega(v_1)\rangle \\
&= \langle \Omega(\tilde{Y}L_{\bar{\mu}}S(\tilde{X})), \Omega(XL_{\bar{\lambda}}S(Y))\rangle \\
&= \langle \Omega(\tilde{Y})L_{-\bar{\mu}}S(S^{-2}(\Omega(\tilde{X}))), \Omega(X)L_{-\bar{\lambda}}S(S^{-2}(\Omega(Y)))\rangle \\
&= \sqrt{\chi}(\mu,-\lambda)(\Omega(\tilde{Y}),X)(S^{-2}(\Omega(Y)),\Omega(S^{-2}(\Omega(\tilde{X})))) \\
&= \sqrt{\chi}(\mu,-\lambda)(\Omega(\tilde{Y}),X)(S^{-2}(\Omega(Y)),S^2(\tilde{X})) \\
&= \sqrt{\chi}(\mu,-\lambda)(\Omega(\tilde{Y}),X)(\Omega(Y),\tilde{X}) \\
&= \langle v_1, v_2\rangle.
\end{aligned}
$$

We have

$$\begin{aligned}
&\Omega(\mathrm{ad}(E_i)\cdot v_1)\\
&= \Omega(E_iv_1 - L_{\overline{\alpha_i}}v_1L_{-\overline{\alpha_i}}E_i)\\
&= \Omega(E_iv_1 - \chi(\alpha_i,\theta-\gamma)v_1E_i)\\
&= -\chi(\alpha_i,\theta-\gamma)(-E_i^\vee L_{\overline{\alpha_i}}\Omega(v_1)L_{-\overline{\alpha_i}} + \Omega(v_1)E_i^\vee)\\
&= -\chi(\alpha_i,\theta-\gamma)\Omega(v_1)\cdot\widetilde{\mathrm{ad}}(E_i^\vee),
\end{aligned}$$

and

$$\begin{aligned}
&\Omega(v_2\cdot\widetilde{\mathrm{ad}}(E_i))\\
&= \Omega(-L_{-\overline{\alpha_i}}E_iv_2 + L_{-\overline{\alpha_i}}v_2E_i)\\
&= \chi(-\alpha_i,\omega-\delta+\alpha_i)\Omega(-E_iv_2L_{-\overline{\alpha_i}} + v_2E_iL_{-\overline{\alpha_i}})\\
&= -\chi(-\alpha_i,\omega-\delta+\alpha_i)(E_i^\vee\Omega(v_2)L_{\overline{\alpha_i}} - \Omega(v_2)E_i^\vee L_{\overline{\alpha_i}})\\
&= -\chi(-\alpha_i,\omega-\delta+\alpha_i)\mathrm{ad}(E_i^\vee)\cdot\Omega(v_2).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\langle\mathrm{ad}(E_i^\vee)\cdot\Omega(v_2),\Omega(v_1)\rangle\\
&= -\chi(\alpha_i,\omega-\delta+\alpha_i)\langle\Omega(v_2\cdot\widetilde{\mathrm{ad}}(E_i)),\Omega(v_1)\rangle\\
&= -\chi(\alpha_i,\omega-\delta+\alpha_i)\langle v_1, v_2\cdot\widetilde{\mathrm{ad}}(E_i)\rangle\\
&= -\chi(\alpha_i,\omega-\delta+\alpha_i)\langle\mathrm{ad}(E_i)\cdot v_1, v_2\rangle\\
&= -\chi(\alpha_i,\omega-\delta+\alpha_i)\langle\Omega(v_2),\Omega(\mathrm{ad}(E_i)\cdot v_1)\rangle\\
&= \chi(\alpha_i,\omega-\delta+\alpha_i+\theta-\gamma)\langle\Omega(v_2),\Omega(v_1)\cdot\widetilde{\mathrm{ad}}(E_i^\vee))\rangle\\
&= \chi(\alpha_i,\omega-\delta+\alpha_i+\theta-\gamma)\delta_{-(-(\omega-\delta)),-(\theta-\gamma)-\alpha_i}\langle\Omega(v_2),\Omega(v_1)\cdot\widetilde{\mathrm{ad}}(E_i^\vee))\rangle\\
&= \delta_{-(-(\omega-\delta)),-(\theta-\gamma)-\alpha_i}\langle\Omega(v_2),\Omega(v_1)\cdot\widetilde{\mathrm{ad}}(E_i^\vee))\rangle\\
&= \langle\Omega(v_2),\Omega(v_1)\cdot\widetilde{\mathrm{ad}}(E_i^\vee)\rangle,
\end{aligned}$$

as desired. This completes the proof. □

4 Harish-Chandra map

We define the $\mathbb{C}$-linear map $\Phi : U \to U^0$ by $\Phi(X^\vee L_{\bar\mu}X) := \varepsilon(X^\vee)\varepsilon(X)L_{\bar\mu}$ for all $X \in U^+$, all $\mu \in \mathbb{Z}\Pi'$, and all $X^\vee \in U^-$. For $\lambda \in \mathbb{Z}\Pi'$, define $\sqrt{\chi}_{\bar\lambda} \in (U^0)^*$ by $\sqrt{\chi}_{\bar\lambda}(L_{\bar\mu}) := \sqrt{\chi}(\lambda,\mu)$ for all $\mu \in \mathbb{Z}\Pi'$. Define the $\mathbb{C}$-linear monomorphism $\zeta : U \to U^*$ by $\zeta(v_2)(v_1) := \langle v_1, v_2\rangle$. Define the right action of U on U^* by $f\cdot u(v) := f(\mathrm{ad}(u)\cdot v)$ for and $f \in U^*$, all u, $v \in u$. By (3.7), we have $\zeta(v)\cdot u = \zeta(v\cdot\widetilde{\mathrm{ad}}(u))$ for all u, $v \in u$. Let $\mathfrak{z}(U)$ be the center of U, that is $\mathfrak{z}(U) := \{u \in U | \forall v \in U,\ uv = vu\}$. It is easy to see that $\mathfrak{z}(U) = \{u \in U | \forall v \in U,\ u\cdot\widetilde{\mathrm{ad}}(v) = \varepsilon(v)u\}$.

$$\text{(4.1)} \qquad \text{Assume that } \exists\ \tilde{\rho} \in \mathbb{Z}\Pi',\ \forall i \in J_{1,\ell},\ \chi(\tilde{\rho}, \alpha_i) = \chi(\alpha_i, \alpha_i).$$

Then $S^2(u) = L_{-\bar{\tilde{\rho}}} u L_{\bar{\tilde{\rho}}}$ hold for all $u \in U$.

Let V be a finite dimensional left U-module. Define $f_V \in U^*$ by $f_V(u) := \mathrm{Tr}(uL_{-\bar{\tilde{\rho}}}; V)$. Then for all u, $v \in U$ with $\Delta(u) = \sum_r u_r^{(1)} \otimes u_r^{(2)}$ we have

$$\begin{aligned}
&(f_V \cdot u)(v) = f_V(\mathrm{ad}(u) \cdot v) \\
&= \sum_r f_V(u_r^{(1)} v S(u_r^{(2)})) = \sum_r \mathrm{Tr}(u_r^{(1)} v S(u_r^{(2)}) L_{-\bar{\tilde{\rho}}}; V) \\
&= \sum_r \mathrm{Tr}(v S(u_r^{(2)}) L_{-\bar{\tilde{\rho}}} u_r^{(1)}; V) = \sum_r \mathrm{Tr}(v S(u_r^{(2)}) S^2(u_r^{(1)}) L_{-\bar{\tilde{\rho}}}; V) \\
&= \sum_r \mathrm{Tr}(v S(S(u_r^{(1)}) u_r^{(2)}) L_{-\bar{\tilde{\rho}}}; V) = \mathrm{Tr}(v S(\varepsilon(u)) L_{-\bar{\tilde{\rho}}}; V) \\
&= \mathrm{Tr}(\varepsilon(u) v L_{-\bar{\tilde{\rho}}}; V) = \varepsilon(u) f_V(v),
\end{aligned}$$

so we have $f_V \cdot u = \varepsilon(u) f_V$. Hence we see that

$$\text{(4.2)} \qquad f_V \in \mathrm{Im}(\zeta) \quad \Longrightarrow \quad \zeta^{-1}(f_V) \in \mathfrak{z}(U).$$

$$\text{(4.3)} \qquad \begin{array}{c} \text{Assume that } \exists \lambda \in \mathbb{Z}\Pi',\ \exists v_{\bar{\lambda}} \in V \setminus \{0\},\ V = \oplus_{\beta \in \mathbf{Z}_{\geq 0}\Pi} U^-_{-\beta} v_{\bar{\lambda}}, \\ \forall \mu \in \mathbb{Z}\Pi',\ L_{\bar{\mu}} v_{\bar{\lambda}} = \sqrt{\chi}(\mu, \lambda) v_{\bar{\lambda}},\ \forall i \in J_{1,\ell},\ E_i v_{\bar{\lambda}} = 0. \end{array}$$

Lemma 5. *We have $f_V \in \mathrm{Im}(\zeta)$. In particular, $\zeta^{-1}(f_V) \in \mathfrak{z}(U)$. Further we have*

$$\text{(4.4)} \qquad \Phi(\zeta^{-1}(f_V)) = \sum_{\beta \in \mathbf{Z}_{\geq 0}\Pi} (\dim U^-_{-\beta} v_{\bar{\lambda}}) \sqrt{\chi}(\tilde{\rho}, 2\beta - \lambda) L_{\overline{2\beta - \lambda}},$$

and

$$\begin{aligned}
&\zeta^{-1}(f_V) - \Phi(\zeta^{-1}(f_V)) \\
&\in \sum_{\omega \in \mathbf{Z}_{\geq 0}\Pi \setminus \{0\}, \beta \in \mathbf{Z}_{\geq 0}\Pi, \dim U^-_{-\beta-\omega} v_{\bar{\lambda}} \neq 0} \mathrm{Span}_{\mathbb{C}}(U^-_{-\omega} L_{\overline{\omega + 2\beta - \lambda}} U^+_{\omega}).
\end{aligned}$$

Proof. Let $v_1 = X L_{\bar{\nu}} Y$, $v_2 = \tilde{Y} L_{\bar{\mu}} \tilde{X} \in U$ with ν, $\mu \in \mathbb{Z}\Pi'$, $X \in U^+_{\theta}$, $Y \in U^-_{-\gamma}$,

$\tilde{X} \in U^+_\omega$, $\tilde{Y} \in U^-_{-\delta}$, and θ, γ, ω, $\delta \in \mathbb{Z}_{\geq 0}\Pi$. Then we have

$$
\begin{aligned}
&\zeta(v_2)(v_1) = \langle v_1, v_2\rangle = \langle XL_{\bar{\nu}}Y, \tilde{Y}L_{\bar{\mu}}\tilde{X}\rangle \\
&= \langle XL_{\bar{\nu}}S(S^{-1}(Y)L_{-\bar{\gamma}}L_{\bar{\gamma}}), \tilde{Y}L_{\bar{\mu}}S(S^{-1}(\tilde{X})L_{\bar{\omega}}L_{-\bar{\omega}})\rangle \\
&= \langle XL_{\overline{\nu-\gamma}}S(S^{-1}(Y)L_{-\bar{\gamma}}), \tilde{Y}L_{\overline{\mu+\omega}}S(S^{-1}(\tilde{X})L_{\bar{\omega}})\rangle \\
&= \sqrt{\chi}(-\nu+\gamma, \mu+\omega)(X, \Omega(\tilde{Y}))(S^{-1}(\tilde{X})L_{\bar{\omega}}, \Omega(S^{-1}(Y)L_{-\bar{\gamma}})) \\
&= \sqrt{\chi}(-\nu+\gamma, \mu+\omega)(X, \Omega(\tilde{Y}))(S^{-1}(L_{-\bar{\omega}}\tilde{X}), \Omega(S^{-1}(L_{\bar{\gamma}}Y))) \\
&= \sqrt{\chi}(-\nu+\gamma, \mu+\omega)(X, \Omega(\tilde{Y}))(S^{-1}(L_{-\bar{\omega}}\tilde{X}), S(\Omega(L_{\bar{\gamma}}Y))) \\
&= \sqrt{\chi}(-\nu+\gamma, \mu+\omega)(X, \Omega(\tilde{Y}))(S(S^{-1}(L_{-\bar{\omega}}\tilde{X})), \Omega(L_{\bar{\gamma}}Y)) \\
&= \sqrt{\chi}(-\nu+\gamma, \mu+\omega)\chi(-\omega,\omega)\chi(-\gamma,\gamma)(X, \Omega(\tilde{Y}))(\tilde{X}L_{-\bar{\omega}}, \Omega(Y)L_{-\bar{\gamma}}) \\
&= \sqrt{\chi}(-\nu+\gamma, \mu+\omega)\chi(-\omega,\omega)\chi(-\gamma,\gamma)\chi(-\omega,-\gamma)(X, \Omega(\tilde{Y}))(\tilde{X}, \Omega(Y)) \\
&= \sqrt{\chi}(\omega, \mu-\omega)\sqrt{\chi_{\overline{-\mu-\omega}}}(\bar{\nu})\delta_{\theta,\delta}\delta_{\omega,\gamma}(X, \Omega(\tilde{Y}))(\tilde{X}, \Omega(Y)).
\end{aligned}
$$

Let $\beta \in \mathbb{Z}_{\geq 0}\Pi$. Let $m_\beta := \dim U^+_\beta$. Let $X_{\beta,x} \in U^+_\beta$ and $Y_{-\beta,x} \in U^-_{-\beta}$ ($x \in J_{1,m_\beta}$) be such that $(X_{\beta,x}, \Omega(Y_{-\beta,y})) = \delta_{xy}$. Then $\{X_{-\beta,x}|x \in J_{1,m_\beta}\}$ and $\{Y_{-\beta,x}|x \in J_{1,m_\beta}\}$ is $\mathbb{C}$-bases of U^+_β and $U^-_{-\beta}$ respectivly. Let $k_\beta := \dim U^-_{-\beta}v_{\bar{\lambda}}$. Let $\{Z_{-\beta,r}v_{\bar{\lambda}}|r \in J_{1,k_\beta}\}$ be a $\mathbb{C}$-basis of $U^-_{-\beta}v_{\bar{\lambda}}$. For $r \in J_{1,k_\beta}$, Define $t_{-\beta,r} \in V^*$ by $t_{-\beta,r}(Z_{-\beta',r'}v_{\bar{\lambda}}) := \delta_{\beta,\beta'}\delta_{r,r'}$.

Let $v_1 \in U$ be as above. Then we have

$$
\begin{aligned}
f_V(v_1) &= \sum_{\beta\in\mathbb{Z}_{\geq 0}\Pi, r\in J_{1,k_\beta}} t_{-\beta,r}(v_1 L_{-\bar{\tilde{\rho}}} Z_{-\beta,r}v_{\bar{\lambda}}) \\
&= \sum_{\beta,r} \chi(-\tilde{\rho},-\beta)\sqrt{\chi}(-\tilde{\rho},\lambda)t_{-\beta,r}(v_1 Z_{-\beta,r}v_{\bar{\lambda}}) \\
&= \sum_{\beta,r} \chi(-\tilde{\rho},-\beta)\sqrt{\chi}(-\tilde{\rho},\lambda)t_{-\beta,r}(XL_{\bar{\nu}}YZ_{-\beta,r}v_{\bar{\lambda}}) \\
&= \sum_{\beta,r} \chi(-\tilde{\rho},-\beta)\sqrt{\chi}(-\tilde{\rho},\lambda)\chi(\nu,-\gamma-\beta)\sqrt{\chi}(\nu,\lambda)t_{-\beta,r}(XYZ_{-\beta,r}v_{\bar{\lambda}}) \\
&= \sum_{\beta,r} \sqrt{\chi}(\tilde{\rho}, 2\beta-\lambda)\sqrt{\chi_{\overline{-2\gamma-2\beta+\lambda}}}(\bar{\nu})\delta_{\theta,\gamma}t_{-\beta,r}(XYZ_{-\beta,r}v_{\bar{\lambda}}).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
f_V \;=\; & \sum_{\beta\in\mathbb{Z}_{\geq 0}\Pi, r\in J_{1,k_\beta}} \; \sum_{\omega\in\mathbb{Z}_{\geq 0}\Pi, x,y\in J_{1,m_\beta}} \\
& \cdot\sqrt{\chi}(\tilde{\rho}-\omega, 2\beta-\lambda)t_{-\beta,r}(X_{\omega,x}Y_{-\omega,y}Z_{-\beta,r}v_{\bar{\lambda}}) \\
& \cdot\zeta(Y_{-\omega,y}L_{\overline{\omega+2\beta-\lambda}}X_{\omega,x}),
\end{aligned}
$$

as desired. $\square$

Let $\lambda \in \mathbb{Z}\Pi'$. Let $M(\lambda)$ be the left U-module satisfying (4.3) and satisfying that $\dim U^-_{-\beta} v_{\bar{\lambda}} = \dim U^-_{-\beta}$ for all $\beta \in \mathbb{Z}_{\geq 0}\Pi$. Let $I(\lambda)$ be the proper ideal of $M(\lambda)$ defined as the sum of proper ideals I' of $M(\lambda)$ with $I' \subset \oplus_{\beta \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}} U^+_\beta v_{\bar{\lambda}}$. Let $V(\lambda) := M(\lambda)/I(\lambda)$. Note that $V(\lambda)$ satisfies (4.3).

(4.5) For $r \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{C}$, let $\{r\}_t := \sum_{k \in J_{1,r}} t^{k-1}$, and $\{r\}_t! := \prod_{k \in J_{1,r}} \{k\}_t$.

(4.6) For $i \in J_{1,\ell}$, let $q_i := \sqrt{\chi}(\alpha_i, \alpha_i)$, so $q_i^2 := \chi(\alpha_i, \alpha_i)$.

Let $i \in J_{1,\ell}$ and $r \in \mathbb{Z}_{\geq 0}$. Then we have

$$\begin{aligned}(E_i^r, E_i^r) &= \sum_{k \in J_{1,r}} (E_i^{k-1} L_{\overline{\alpha_i}} E_i^{r-k}, E_i^{r-1}) = \{r\}_{q_i^2} (E_i^{r-1} L_{\overline{\alpha_i}}, E_i^{r-1}) \\ &= \{r\}_{q_i^2} (E_i^{r-1}, E_i^{r-1}) = \{r\}_{q_i^2}!,\end{aligned}$$

which implies

$$\{r\}_{q_i^2}! = 0 \Leftrightarrow E_i^r = 0 \Leftrightarrow (E_i^\vee)^r = 0, \tag{4.7}$$

since $E_i^\vee = \Omega(E_i)$. We also have

$$\begin{aligned}&E_i (E_i^\vee)^r - (E_i^\vee)^r E_i \\ &= \sum_{k \in J_{1,r}} (E_i^\vee)^{r-k} (-L_{\overline{\alpha_i}} + L_{-\overline{\alpha_i}}) (E_i^\vee)^{k-1} \\ &= (E_i^\vee)^{r-1} \sum_{k \in J_{1,r}} (-q_i^{-2(k-1)} L_{\overline{\alpha_i}} + q_i^{2(k-1)} L_{-\overline{\alpha_i}}) \\ &= \{r\}_{q_1^2} (E_i^\vee)^{r-1} (-q_i^{-2(r-1)} L_{\overline{\alpha_i}} + L_{-\overline{\alpha_i}}).\end{aligned} \tag{4.8}$$

Applying Ω, we have

$$E_i^r E_i^\vee - E_i^\vee E_i^r = \{r\}_{q_i^2} (-q_i^{-2(r-1)} L_{\overline{\alpha_i}} + L_{-\overline{\alpha_i}}) E_i^{r-1}. \tag{4.9}$$

By (4.8), we have

$$\begin{aligned}&E_i (E_i^\vee)^r v_{\bar{\lambda}} \\ &= \{r\}_{q_i^2} (E_i^\vee)^{r-1} (-q_i^{-2(r-1)} \sqrt{\chi}(\alpha_i, \lambda) + \sqrt{\chi}(\alpha_i, -\lambda)) v_{\bar{\lambda}} \\ &= -\{r\}_{q_i^2} \sqrt{\chi}(\alpha_i, -\lambda) (\chi(\alpha_i, \lambda) q_i^{-2(r-1)} - 1) (E_i^\vee)^{r-1} v_{\bar{\lambda}}.\end{aligned} \tag{4.10}$$

5 Rank one case

In this section, we assume that $\ell = 1$. Let V be a U-module satisfying (4.3). Let $r \in \mathbb{N}$ be such that

(5.1) $$(\chi(\alpha_1,\lambda)q_1^{-2(r-1)} - 1)\{r\}_{q_1^2} = 0,\ (E_1^\vee)^{r-1}v_{\bar\lambda} \neq 0 \text{ and } (E_1^\vee)^r v_{\bar\lambda} = 0.$$

Then $\dim V = r$. We have

(5.2)
$$\begin{aligned}
&\zeta^{-1}(f_V)\\
&= \sum_{k\in J_{0,r-1}, m\in J_{0,r-1-k}} \sqrt{\chi}(\tilde\rho - m\alpha_1, 2k\alpha_1 - \lambda)\\
&\qquad \cdot t_{-\beta,r}(E_1^m(\frac{1}{\{m\}_{q_1^2}!}(E_1^\vee)^m)(E_1^\vee)^k v_{\bar\lambda})\\
&\qquad \cdot \frac{1}{\{m\}_{q_1^2}!}(E_1^\vee)^m L_{\overline{(m+2k)\alpha_1-\lambda}} E_1^m\\
&= \sqrt{\chi}(\tilde\rho, -\lambda) \sum_{k\in J_{0,r-1}, m\in J_{0,r-1-k}} \sqrt{\chi}(\alpha_1,\lambda)^m \frac{q_1^{2(1-m)k}}{(\{m\}_{q_1^2}!)^2}\\
&\qquad \cdot(-1)^m(\prod_{t\in J_{1,m}}(\chi(\alpha_1,\lambda)q_1^{-2(m+k-t)} - 1))\\
&\qquad \cdot\sqrt{\chi}(\alpha_1,-\lambda)^m \frac{\{m+k\}_{q_1^2}!}{\{k\}_{q_1^2}!}(E_1^\vee)^m L_{\overline{(m+2k)\alpha_1-\lambda}} E_1^m\\
&= \sqrt{\chi}(\tilde\rho, -\lambda) \sum_{m\in J_{0,r-1}} (E_1^\vee)^m \Big((-1)^m \sum_{k\in J_{0,r-m-1}} \frac{q_1^{2(1-m)k}\{m+k\}_{q_1^2}!}{(\{m\}_{q_1^2}!)^2\{k\}_{q_1^2}!}\\
&\qquad \cdot(\prod_{t\in J_{1,m}}(\chi(\alpha_1,\lambda)q_1^{-2(m+k-t)} - 1))L_{\overline{(m+2k)\alpha_1-\lambda}}\Big) E_1^m,
\end{aligned}$$

which implies

(5.3) $$\Phi(\zeta^{-1}(f_V)) = \sqrt{\chi}(\tilde\rho, -\lambda) \sum_{k\in J_{0,r-1}} q_1^{2k} L_{\overline{2k\alpha_1-\lambda}}.$$

If $\{r\}_{q_1^2} \neq 0$, then $\chi(\alpha_1, \lambda) = q_1^{2(r-1)}$, which implies

(5.4)
$$\begin{aligned}
&\zeta^{-1}(f_V) \\
&= \sqrt{\chi}(\tilde{\rho}, -\lambda) L_{\overline{(r-1)\alpha_1 - \lambda}} \\
&\qquad \cdot \sum_{m \in J_{0,r-1}} (E_1^{\vee})^m \Big((-1)^m \sum_{k \in J_{0,r-m-1}} \frac{q_1^{2(1-m)k} \{m+k\}_{q_1^2}!}{(\{m\}_{q_1^2}!)^2 \{k\}_{q_1^2}!} \\
&\qquad \cdot \big(\prod_{t \in J_{1,m}} (q_1^{2((r-1)-(m+k)+t)} - 1) \big) L_{\overline{(m+2k-r+1)\alpha_1}} \Big) E_1^m,
\end{aligned}$$

which implies

$$\Phi(\zeta^{-1}(f_V)) = L_{\overline{(r-1)\alpha_1 - \lambda}} \sqrt{\chi}(\tilde{\rho}, -\lambda) \sum_{k \in J_{0,r-1}} q_1^{2k} L_{\overline{(2k-r+1)\alpha_1}}. \tag{5.5}$$

Theorem 6. *Assume that $\ell = 1$ and $q_1^2 \neq 1$. Let $\{\nu_p \in \mathbb{Z}\Pi' | p \in P\}$ be a set of representatives of $\{\bar{\nu} \in \overline{\mathbb{Z}\Pi'} | \nu \in \mathbb{Z}\Pi', \chi(\alpha_1, \nu) = 1\}$. Assume that for all $x \in \mathbb{N}$, $x\overline{\alpha_1} \neq \bar{0}$, so $L_{\overline{\alpha_1}}^x \neq 1$.*

(1) *Assume that $\{k\}_{q_1^2}! \neq 0$ for all $k \in \mathbb{N}$. Then*

$$\mathfrak{z}(U) = \bigoplus_{p \in P, k \in \mathbb{Z}_{\geq 0}} \mathbb{C}\zeta^{-1}(f_{V(k\alpha_1)}) L_{\overline{\nu_p}}, \tag{5.6}$$

as a $\mathbb{C}$-linear space. In particular, as a $\mathbb{C}$-algebra, $\mathfrak{z}(U)$ is generated by $L_{\overline{\nu_p}}$ ($p \in P$), and $-(q_1^2 - 1)E_1^{\vee} E_1 + L_{-\overline{\alpha_1}} + q_1^2 L_{\overline{\alpha_1}}$.

(2) *Assume that there exists $r \in \mathbb{N}$ such that $\{r-1\}_{q_1^2}! \neq 0$ and $\{r\}_{q_1^2} = 0$. Let $\mathcal{R} := \{\bar{\mu} \in \overline{\mathbb{Z}\Pi'} | \mu \in \mathbb{Z}\Pi', \chi(\alpha_1, \mu) \notin \{q_1^s | s \in J_{0,r-2}\}\}$. Then we have*

$$\mathfrak{z}(U) = \Big(\bigoplus_{p \in P, k \in J_{0,r-2}} \mathbb{C}\zeta^{-1}(f_{V(k\alpha_1)}) L_{\overline{\nu_p}} \Big) \oplus \Big(\bigoplus_{\eta \in \mathcal{R}} \mathbb{C}\zeta^{-1}(f_{V(\eta)}) \Big), \tag{5.7}$$

as a $\mathbb{C}$-linear space, where let $V(\eta) := V(\bar{\mu})$ if $\eta = \bar{\mu}$.

Proof. Let $C := \sum_{m \in J_{0,k}} (E_1^{\vee})^m z_m E_1^m \in \mathfrak{z}(U)$ with $z_m \in U^0$. Define the $\mathbb{C}$-algebra automorphism $g : U^0 \to U^0$ by $f(L_{\bar{\lambda}}) = \chi(\alpha_1, -\lambda) L_{\bar{\lambda}}$ for all $\lambda \in \mathbb{Z}\Pi'$. By

(4.8)-(4.9), we have

(5.8)
$$
\begin{aligned}
0 &= CE_1 - E_1C \\
&= \sum_{m\in J_{0,k}} ((E_1^\vee)^m (z_m - g(z_m))E_1^{m+1} \\
&\qquad -(E_1^\vee)^{m-1}\{m\}_{q_1^2}(-q_1^{-2(m-1)}L_{\overline{\alpha_1}} + L_{-\overline{\alpha_1}}z_m)E_1^m) \\
&= \left(\sum_{m\in J_{0,k-1}} (E_1^\vee)^m (z_m - g(z_m) - \{m+1\}_{q_1^2}(-q_1^{-2m}L_{\overline{\alpha_1}} + L_{-\overline{\alpha_1}})z_{m+1})E_1^{m+1}\right) \\
&\qquad +E_1^{k+1}(z_k - g(z_k))(E_1^\vee)^k,
\end{aligned}
$$

and

(5.9)
$$
\begin{aligned}
0 &= E_1^\vee C - CE_1^\vee \\
&= \sum_{m\in J_{0,k}} ((E_1^\vee)^{m+1} (z_m - g(z_m))E_1^{m} \\
&\qquad -(E_1^\vee)^{m}\{m\}_{q_1^2}(-q_1^{-2(m-1)}L_{\overline{\alpha_1}} + L_{-\overline{\alpha_1}})z_mE_1^{m-1}) \\
&= \left(\sum_{m\in J_{0,k-1}} (E_1^\vee)^{m+1} (z_m - g(z_m) - \{m+1\}_{q_1^2}(-q_1^{-2m}L_{\overline{\alpha_1}} + L_{-\overline{\alpha_1}})z_{m+1})E_1^{m}\right) \\
&\qquad +E_1^{k}(z_k - g(z_k))(E_1^\vee)^{k+1}.
\end{aligned}
$$

Hence we have

(5.10) $$E_1^{k+1} \neq 0 \Rightarrow z_k = g(z_k),$$

and

(5.11) $$\forall m' \in J_{0,k-1}, \quad E_1^{m'} \neq 0 \Rightarrow z_{m'} - g(z_{m'}) = \{m'+1\}_{q_1^2}(-q_1^{-2m'}L_{\overline{\alpha_1}} + L_{-\overline{\alpha_1}}).$$

(1) Let C be as above. By (5.10), we have $z_k \in \oplus_{p\in P}\mathbb{C}L_{\overline{\nu_p}}$. By (5.4), the last term of $\zeta^{-1}(f_{V(k\alpha_1)})$ is $b(E_1^\vee)^k E_1^k$ for some $b \in \mathbb{C}^\times$. Then we can see (5.6).

(2) Let C be as above and assume that $k = r-1$. Assume that C is not in RHS of (5.7). By (5.8)-(5.9), we may assume that there exists $\lambda \in \mathbb{Z}\Pi'$ such that $z_m \in \oplus_{y\in\mathbb{Z}}\mathbb{C}L_{\bar\lambda+2y\overline{\alpha_1}}$ for all $m \in J_{0,r-1}$. By the same argument as in (1), we may assume $z_{r-1} \neq 0$. For $\mu \in \mathbb{Z}\Pi'$ with $\bar\mu \notin \mathcal{R}$, $V(\mu) = M(\mu)$ and, by (5.2), for $\mu \in \mathbb{Z}\Pi'$, the last term of $\zeta^{-1}(f_{M(\mu)})$ is $c(E_1^\vee)^{r-1}L_{\overline{(r-1)\alpha_1-\mu}}E_1^{r-1}$ for some $c \in \mathbb{C}^\times$. Hence we may assume that $\chi(\lambda, \alpha_1) = 1$ and $z_{r-1} \in \oplus_{x\in\mathbb{Z}_{\geq 0}, y\in J_{1,r-1}}\mathbb{C}L_{\bar\lambda+(rx+y)\overline{\alpha_1}}$. However this contradicts (5.11) since $z' - g(z') \in \oplus_{x_1\in\mathbb{Z}, y_1\in J_{1,r-1}}\mathbb{C}L_{\bar\lambda+(rx_1+y_1)\overline{\alpha_1}}$ for all $z' \in \oplus_{x_2\in\mathbb{Z}}\mathbb{C}L_{\bar\lambda+rx_2\overline{\alpha_1}}$. □

6 Higher rank case

Assume that $\ell \in \mathbb{N}$. From now on,

$$\text{assume that } q_i^2 \neq 1 \text{ for all } i \in J_{1,\ell}, \tag{6.1}$$

and

$$\begin{gathered}\text{assume that there exist } \omega_i \in \mathbb{Z}\Pi' \ (i \in J_{1,\ell}) \text{ such that} \\ \sqrt{\chi}(\omega_i, \alpha_i)^{r_i} \neq 1 \text{ for all } r_i \in \mathbb{N} \text{ and } \sqrt{\chi}(\omega_i, \alpha_j) = 1 \text{ for } i \neq j.\end{gathered} \tag{6.2}$$

Then $L_{\bar{\gamma}} \neq 1$ for all $\bar{\gamma} \in \mathbb{Z}\Pi \setminus \{0\}$. Further

$$\mathfrak{z}(U) \subset \bigoplus_{\beta \in \mathbb{Z}_{\geq 0}\Pi} \mathrm{Span}_{\mathbb{C}}(U^-_{-\beta} U^0 U^+_\beta). \tag{6.3}$$

Moreover we have

Lemma 7. *Let $z \in \mathfrak{z}(U)$. Assume that $\Phi(z) = 0$. Then $z = 0$.*

Proof. Let $\beta \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$. Let $X_r \in U^+_\beta$, and $Y_r \in U^-_{-\beta}$ $(r \in J_{1,\dim U^+_\beta})$ be $\mathbb{C}$-base elelemnts of U^+_β, and $U^-_{-\beta}$ respectively such that $(X_r, \Omega(Y_k)) = \delta_{rk}$. By (2.14), and formulas similar to (3.10), we have

$$\Phi(X_r Y_k) \in \delta_{rk} L_{-\bar{\beta}} + \sum_{\bar{\gamma} \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}} \mathbb{C} L_{-\bar{\beta}+\bar{\gamma}}. \tag{6.4}$$

Then we can easily see that the statemet holds. □

Let $i \in J_{1,\ell}$. Let $(\mathbb{Z}\Pi')_i := \{\lambda \in \mathbb{Z}\Pi' | \exists t \in \mathbb{Z},\ \chi(\alpha_i, \lambda) = q_i^{2t}\}$, and $\overline{(\mathbb{Z}\Pi')_i} := \{\bar{\lambda} \in \mathbb{Z}\Pi'/T | \lambda \in (\mathbb{Z}\Pi')_i\}$. If $q_i^m \neq 1$ for all $m \in \mathbb{N}$, define the map $\sigma_i : (\mathbb{Z}\Pi')_i \to (\mathbb{Z}\Pi')_i$ by letting $\sigma_i(\lambda)$ be such that $\sigma_i(\lambda) \in \lambda + \mathbb{Z}\alpha_i$ and $\chi(\lambda + \sigma_i(\lambda), \alpha_i) = 1$ for all $\lambda \in (\mathbb{Z}\Pi')_i$; we also denote the map $\overline{(\mathbb{Z}\Pi')_i} \to \overline{(\mathbb{Z}\Pi')_i}$ induced from σ_i by the same symbol. Assume that there exists $r \in \mathbb{N}$ such that $\{r-1\}_{q_i^2}! \neq 0$ and $\{r\}_{q_i^2} = 0$. Define the map $\tau_i : (\mathbb{Z}\Pi')_i \to (\mathbb{Z}\Pi')_i$ as follows. Let $\lambda \in (\mathbb{Z}\Pi')_i$. Let $y \in J_{0,r-1}$ be such that $\chi(\lambda + y\alpha_i, \alpha_i) = 1$. Then we let $\tau_i(\lambda) := \lambda + 2y\alpha_i$. We also denote the map $\overline{(\mathbb{Z}\Pi')_i} \to \overline{(\mathbb{Z}\Pi')_i}$ induced from τ_i by the same symbol.

Theorem 8. *Let $i \in J_{1,\ell}$. Let $\sum_{\eta \in \overline{\mathbb{Z}\Pi'}} a_\eta L_\eta \in \Phi(\mathfrak{z}(U))$ with $a_\eta \in \mathbb{C}$.*

(1) *Assume that $q_i^m \neq 1$ for all $m \in \mathbb{N}$. Then we have $\Phi(\mathfrak{z}(U)) \subset \oplus_{\eta \in \overline{(\mathbb{Z}\Pi')_i}} \mathbb{C} L_\eta$. Further we have*

$$\forall \eta_1 \in \overline{(\mathbb{Z}\Pi')_i},\ \sqrt{\chi}_{\bar{\rho}}(-\sigma_i(\eta_1)) a_{\sigma_i(\eta_1)} = \sqrt{\chi}_{\bar{\rho}}(-\eta_1) a_{\eta_1}. \tag{6.5}$$

(2) *Assume that there exists $r \in \mathbb{N}$ such that $\{r-1\}_{q_i^2}! \neq 0$ and $\{r\}_{q_i^2} = 0$. Then for $\lambda \in \mathbb{Z}\Pi' \setminus (\mathbb{Z}\Pi')_i$, we have*

$$\forall m \in \{0\} \cup J_{2,r-1},\ \sum_{k \in \mathbb{Z}} q_i^{2mk} a_{\bar{\lambda} + 2k\overline{\alpha_i}} = 0. \tag{6.6}$$

Further for $\mu \in \mathbb{Z}\Pi'$ *with* $\chi(\mu, \alpha_i) = q_i^{2d}$ *for some* $d \in J_{1,r-1}$*, we have*

$$\sum_{k\in\mathbb{Z}} (-q_i^{-2d})^k a_{\tau_i^k(\bar{\mu})} = 0. \tag{6.7}$$

Proof. Let $z = \sum_{\beta\in\mathbb{Z}_{\geq 0}\Pi} z_\beta \in \mathfrak{z}(U)$ with $z_\beta \in \mathrm{Span}_{\mathbb{C}}(U^-_{-\beta}U^0U^+_\beta)$. Let $i \in J_{1,\ell}$. We see that $E_i(z_0 + z_{\alpha_i}) - (z_0 + z_{\alpha_i})E_i = E_i^\vee(z_0 + z_{\alpha_i}) - (z_0 + z_{\alpha_i})E_i^\vee = 0$. Then this theorem follows from Theorem 6, and (5.3), (5.5). □

7 $U_q(\mathfrak{gl}(2|1))$

Assume that $N = 4$ and $\ell = 2$. Let $p_1 := p_2 := 0$, and let $p_3 := 1$. Define the symmetric bi-additive map $((\,,\,)) : \mathbb{Z}\Pi' \times \mathbb{Z}\Pi' \to \mathbb{Z}$ by $((\epsilon_i, \epsilon_j)) := (1 - \delta_{i,4})(1 - \delta_{j,4})(-1)^{\delta_{ij}p_i}$. Assume that $\sqrt{\chi}(\epsilon_4, \epsilon_4) = \sqrt{-1}$, $\sqrt{\chi}(\epsilon_4, \epsilon_r) = \sqrt{\chi}(\epsilon_r, \epsilon_4) = 1$ hold for all $r \in J_{1,3}$, and $\sqrt{\chi}(\epsilon_i, \epsilon_j) = q^{\frac{((\epsilon_i,\epsilon_j))}{2}}$ for all $i, j \in J_{1,3}$. Then there exists an additive group isomorphism $\mathbb{Z}^3 \times (\mathbb{Z}/4\mathbb{Z}) \to \overline{\mathbb{Z}\Pi'}$, $(m_1, m_2, m_3, m_4 + 4\mathbb{Z}) \mapsto \sum_{t\in J_{1,4}} m_t\bar{\epsilon}_t$, where $m_t \in \mathbb{Z}$. Assume that $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2 - \epsilon_3 + \epsilon_4$. We also denote this U by $U_q(\mathfrak{gl}(2|1))$. Let $E_{12}^\vee := E_1^\vee E_2^\vee - qE_2^\vee E_1^\vee$. It is well-known that

$$U^- = \bigoplus_{n_1\in\mathbb{Z}_{\geq 0}, n_2, n_{12}\in J_{0,1}} \mathbb{C}(E_2^\vee)^{n_2}(E_{12}^\vee)^{n_{12}}(E_1^\vee)^{n_1}, \tag{7.1}$$

as a $\mathbb{C}$-linear space.

Let $\lambda = \sum_{y\in J_{1,4}} x_y\epsilon_y \in \mathbb{Z}\Pi'$ with $x_y \in \mathbb{Z}$. Let $k := \frac{x_1 - x_2}{2}$. Assume that $k \in \mathbb{Z}_{\geq 0}$. We have $\chi(\lambda, \alpha_1) = q_1^{2k}$. By (4.10), we have a left U-module $K(\lambda)$ satisfying (4.3) and satisfying:

$$K(\lambda) = \bigoplus_{n_1\in J_{0,k+1}, n_2, n_{12}\in J_{0,1}} \mathbb{C}(E_2^\vee)^{n_2}(E_{12}^\vee)^{n_{12}}(E_1^\vee)^{n_1} v_{\bar{\lambda}}, \tag{7.2}$$

as a $\mathbb{C}$-linear space. By (4.4), we have

$$\begin{aligned}
&\Phi(\zeta^{-1}(f_{K(\lambda)})) \\
&= \sqrt{\chi}(\tilde{\rho}, -\lambda)\Big(\Big(\sum_{m_1\in J_{0,k}} (q_1^{2m_1} L_{-\bar{\lambda}+2m_1\overline{\alpha_1}} + q_1^{2(m+1)} L_{-\bar{\lambda}+2(m_1+1)\overline{\alpha_1}+2\overline{\alpha_2}})\Big) \\
&\qquad -L_{-\bar{\lambda}+2\overline{\alpha_2}} - \Big(\sum_{m_2\in J_{1,k-1}} 2q_1^{2m_1} L_{-\bar{\lambda}+2m_1\overline{\alpha_1}+2\overline{\alpha_1}}\Big) - L_{-\bar{\lambda}+2(k+1)\overline{\alpha_1}+2\overline{\alpha_1}}\Big).
\end{aligned}$$

Note that for $m_1, m_2 \in \mathbb{Z}$, we have $\frac{1}{2}((\alpha_1, -\lambda + 2m_1\alpha_1 + 2m_2\alpha_2)) = -k + 2m_1 - m_2$.

For $k \in \mathbb{Z}$, let $[\mathbb{Z}\Pi']_k := \{\mu \in \mathbb{Z}\Pi' | \frac{1}{2}((\alpha_1, \mu)) = k\}$, and let $\overline{[\mathbb{Z}\Pi']}_k := \{\bar{\nu} \in \overline{\mathbb{Z}\Pi'} | \nu \in [\mathbb{Z}\Pi']_k\}$. By the above argument, we see that for $\mu \in [\mathbb{Z}\Pi']_k$ with $k \in \mathbb{N}$,

we have

$$\varPhi(\zeta^{-1}(f_{K(-\mu+2(k+1)\alpha_1+2\alpha_2)})) \in \mathbb{C}^\times L_{\bar{\mu}} \oplus \bigoplus_{m\in J_{-k,k-1}} \bigoplus_{\eta\in\overline{[\mathbb{Z}\Pi']_m}} \mathbb{C}L_\eta. \tag{7.3}$$

Let $z \in \mathfrak{z}(U)$. By Theorem 8 (1),

$$\varPhi(z) = \sum_{\eta\in\overline{[\mathbb{Z}\Pi']_0}} a_\eta L_\eta + \sum_{k\in\mathbb{N}} \sum_{\omega\in\overline{[\mathbb{Z}\Pi']_k}} a_\omega(L_\omega + q_1^{-2k} L_{\omega-2k\overline{\alpha_1}}), \tag{7.4}$$

where a_η, $a_\omega \in \mathbb{C}$. Assume that $a_\omega = 0$ for all $\omega \in \cup_{k\in\mathbb{N}}\overline{[\mathbb{Z}\Pi']_k}$. By Theorem 8 (2), we see that for $\nu \in [\mathbb{Z}\Pi']_k$, if $a_{\bar{\nu}} \neq 0$, then $\nu = x(\epsilon_1+\epsilon_2-\epsilon_3)+2y\epsilon_4$ for some x, $y \in \mathbb{Z}$. By Lemma 7, we have

Theorem 9. *Let U be as above. Then we have*

$$\begin{aligned}
&\mathfrak{z}(U_q(\mathfrak{gl}(2|1))) \\
&= \bigoplus_{y_1\in\mathbb{Z},y_2\in J_{0,1}} \mathbb{C}L_{y_1(\epsilon_1+\epsilon_2-\epsilon_3)+2y_2\epsilon_4} \\
&\qquad \oplus \bigoplus_{x_1\in\mathbb{Z}_{\geq 0},x_2,x_3\in\mathbb{Z},x_4\in J_{0,3}} \mathbb{C}\zeta^{-1}(f_{K((2x_1+x_2+2)\epsilon_1+x_2\epsilon_2+x_3\epsilon_3+x_4\epsilon_4)}).
\end{aligned}$$

8 Lusztig isomorphisms

In this section we may not assume that $\sqrt{\chi}$ is symmetric. For $n \in \mathbb{N}$ and a, $b \in \mathbb{C}$, let $\{n;a,b\} := a^{n-1}b-1$, and $\{n;a,b\}! = \prod_{J_{1,m}}\{m;a,b\}$. Let $D = D(\sqrt{\chi}) = \mathrm{Span}_{\mathbb{C}}(U^+D^0U^-) = \mathrm{Span}_{\mathbb{C}}(U^-D^0U^+)$ be as above. For $\alpha \in \mathbb{Z}\Pi$, let $D_\alpha := \oplus_{\gamma\in\mathbb{Z}\Pi}\mathrm{Span}_{\mathbb{C}}(U^+_\gamma D^0 U^-_{\alpha-\gamma})$.

For $\alpha \in \mathbb{Z}\Pi$ and $X \in D_\alpha$, define four $\mathbb{C}$-linear map ad^L_+X, ad^L_-X, ad^R_+X, $\mathrm{ad}^R_-X : D \to D$ by letting

$$\begin{aligned}
\mathrm{ad}^L_\pm X(Y) &:= XY - \chi(\pm\alpha,\beta)YX, \\
\mathrm{ad}^R_\pm X(Y) &:= XY - \chi(\beta,\pm\alpha)YX
\end{aligned} \tag{8.1}$$

for all $\beta \in \mathbb{Z}\Pi$ and all $Y \in D_\beta$.

Let $q_{ij} := \chi(\alpha_i,\alpha_j)$ for all i, $j \in J_{1,\ell}$. Then $q_{ii} = q_i^2$. We have assumed that $q_{ii} \neq 1$ for all $i \in J_{1,\ell}$.

Lemma 10. *(see [5]) For i, $j \in J_{1,\ell}$ with $i \neq j$ and m, $n \in \mathbb{Z}_{\geq 0}$, in D, we have*

$$\begin{aligned}
&(\mathrm{ad}^L_+E_i)^m(E_j)(\mathrm{ad}^R_+E_i^\vee)^n(E_j^\vee) - (\mathrm{ad}^R_+E_i^\vee)^n(E_j^\vee)(\mathrm{ad}^L_+E_i)^m(E_j) \\
&= \begin{cases}
(-1)^{m+1}\frac{\{m\}_{q_{ii}}!}{\{m-n\}_{q_{ii}}!}\{m;q_{ii},q_{ij}q_{ji}\}!E_i^{m-n}L_{n\alpha_i+\alpha_j} \\
\textit{if } m > n, \\
(-1)^n\{n\}_{q_{ii}}!\{n;q_{ii},q_{ij}q_{ji}\}!(-L_{n\alpha_i+\alpha_j} + L^\vee_{n\alpha_i+\alpha_j}) \\
\textit{if } m = n, \\
(-1)^n\frac{\{n\}_{q_{ii}}!}{\{n-m\}_{q_{ii}}!}\{n;q_{ii},q_{ij}q_{ji}\}!(E_i^\vee)^{n-m}L^\vee_{m\alpha_i+\alpha_j} \\
\textit{if } m < n.
\end{cases}
\end{aligned} \tag{8.2}$$

Let $\sqrt{\chi}$ be as above. Let $i \in I$. We say that $\sqrt{\chi}$ is *i-finite*, if for any $j \in I \backslash \{i\}$, there exists $n \in \mathbb{N}$ such that $\{n\}_{q_{ii}}!\{n; q_{ii}, q_{ij}q_{ji}\}! = 0$.

Assume that $\sqrt{\chi}$ is i-finite. Define $c_{ij} = c(\alpha_i, \alpha_j) \in \mathbb{Z}$ by

$$c_{ij} := \begin{cases} 2 & \text{if } j = i, \\ -\min\{n \in \mathbb{Z}_{\geq 0} | \{n\}_{q_{ii}}!\{n; q_{ii}, q_{ij}q_{ji}\}! \neq 0\}. & \text{if } j \neq i. \end{cases} \tag{8.3}$$

Let

$$\alpha_j^{i\rhd} := \alpha_j - c_{ij}\alpha_i. \tag{8.4}$$

Then $\alpha_j = \alpha_j^{i\rhd} - c_{ij}\alpha_i^{i\rhd}$. Further we have $c(\alpha_j^{i\rhd}, \alpha_k^{i\rhd}) = c(\alpha_j, \alpha_k)$, and $\alpha_j^{i\rhd i\rhd} = \alpha_j$. Let $\Pi^{i\rhd} := \{\alpha_j^{i\rhd} | j \in J_{1,\ell}\}$, and $(\Pi')^{i\rhd} := \Pi'$, $(\Pi'')^{i\rhd} := \Pi''$. Let $D^{i\rhd}$ the Hopf algebra defined in the same way as for D with $\Pi^{i\rhd}$, $(\Pi')^{i\rhd}$, $(\Pi'')^{i\rhd}$ in place of Π, Π', Π''. Define the $\mathbb{Z}$-module map $s_i : \mathbb{Z}(\Pi'')^{i\rhd} \to \mathbb{Z}\Pi''$ to be the *identity map*. Then $s_i(\alpha_j^{i\rhd}) = \alpha_j - c_{ij}\alpha_i$.

By [5], we have:

Theorem 11. *(see [5]) Assume that $\sqrt{\chi}$ is i-finite. Let $q_{i',j'} := \chi(\alpha_{i'}, \alpha_{j'})$ and $q_{i',j'}^{i\rhd} := \chi(\alpha_{i'}^{i\rhd}, \alpha_{j'}^{i\rhd})$ for all i', $j' \in J_{1,\ell}$. Then there exists a $\mathbb{C}$-algebra isomorphism*

$$T_i : D^{i\rhd} \to D \tag{8.5}$$

such that

$$\begin{aligned} & T_i(L_\alpha) = L_{s_i(\alpha)}, \quad T_i(L_\alpha^\vee) = L_{s_i(\alpha)}^\vee, \\ & T_i(E_i) = E_i^\vee L_{-\alpha_i}^\vee, \quad T_i(E_i^\vee) = L_{-\alpha_i}E_i, \\ & T_i(E_j) = (\mathrm{ad}_+^L E_i)^{-c_{ij}}(E_j), \\ & T_i(E_j^\vee) = \frac{(-1)^{-c_{ij}}}{\{-c_{ij}\}_{q_{ii}}!\{-c_{ij}; q_{ii}, q_{ij}q_{ji}\}!}(\mathrm{ad}_+^R E_i^\vee)^{-c_{ij}}(E_j^\vee), \end{aligned} \tag{8.6}$$

and

$$\begin{aligned} & T_i^{-1}(L_\alpha) = L_{s_i(\alpha)}, \quad T_i^{-1}(L_\alpha^\vee) = L_{s_i(\alpha)}^\vee, \\ & T_i^{-1}(E_i) = L_{-\alpha_i}E_i^\vee, \quad T_i^{-1}(E_i^\vee) = E_i L_{-\alpha_i}^\vee, \\ & T_i^{-1}(E_j) = \frac{(q_{ii}^{i\rhd})^{(-c_{ij})(-c_{ij}-1)}(q_{ij}^{i\rhd}q_{ji}^{i\rhd})^{(-c_{ij})}}{\{-c_{ij}\}_{q_{ii}^{i\rhd}}!\{-c_{ij}; q_{ii}^{i\rhd}, q_{ij}^{i\rhd}q_{ji}^{i\rhd}\}!}(\mathrm{ad}_-^R E_i)^{-c_{ij}}(E_j), \\ & T_i^{-1}(E_j^\vee) = (-1)^{c_{ij}}(\mathrm{ad}_-^L E_i^\vee)^{-c_{ij}}(E_j^\vee), \end{aligned} \tag{8.7}$$

for all $\alpha \in \mathbb{Z}\Pi''$ and all $j \in J_{1,\ell} \setminus \{i\}$.

9 Lusztig isomorphisms and $\Phi(\mathfrak{z}(U^{i\rhd}))$

From now on, we again assume that $\sqrt{\chi}$ is symmetric. Assume that $\sqrt{\chi}$ is i-finite. Let $U^{i\rhd}$ the Hopf algebra defined in the same way as for U with $\Pi^{i\rhd}$, $(\Pi')^{i\rhd}$, $(\Pi'')^{i\rhd}$ in place of Π, Π', Π''. Denote the $\mathbb{C}$-algebra isomorphism $U^{i\rhd} \to U$ induced from $T_i : D^{i\rhd} \to D$ by the same symbol. Denote the $\mathbb{Z}$-module isomorphism $\overline{\mathbb{Z}(\Pi')^{i\rhd}} \to \overline{\mathbb{Z}\Pi'}$ induced from $s_i : \mathbb{Z}\Pi'' \to \mathbb{Z}(\Pi'')^{i\rhd}$ by the same symbol; in this stage, s_i is the identity map.

Theorem 12. *Assume that $\sqrt{\chi}$ is i-finite. Let $Z \in \mathfrak{z}(U^{i\rhd})$. Assume that $\Phi(Z) = \sum_{\theta\in\overline{\mathbb{Z}(\Pi')^{i\rhd}}} b_\theta L_\theta$ with $b_\theta \in \mathbb{C}$. Assume that $\Phi(T_i(Z)) = \sum_{\eta\in\overline{\mathbb{Z}\Pi'}} a_\eta L_\eta$ with $a_\eta \in \mathbb{C}$.*

$$(9.1)\quad r := \begin{cases} \min\{r' \in \mathbb{N} | q_i^{2r'} = 1\} & \textit{if there exists } r'' \in \mathbb{N} \textit{ such that } q_i^{2r''} = 1, \\ 0 & \textit{otherwise,} \end{cases}$$

(cf. (6.1)). Then we have

$$(9.2)\qquad a_{\bar\lambda} = \chi(\alpha_i, \lambda)^{1-r} b_{\overline{s_i(\lambda)}},$$

for all $\lambda \in \mathbb{Z}\Pi'$.

Proof. Assume $\ell = 1$. Let $\lambda \in (\mathbb{Z}\Pi')^{1\rhd}$ and $r \in \mathbb{N}$. Assume that $\prod_{0\in J_{0,r-2}}(\chi(\alpha_1^{1\rhd}, \lambda)q_1^{-2t} - 1) \neq 0$. Assume that $\{r-1\}_{q_1^2}! \neq 0$ and $\{r\}_{q_1^2}! = 0$. Let
(9.3)

$$Y^{1\rhd} := \sqrt{\chi}(\alpha_1^{1\rhd}, -\lambda)(-1)^{r-1}(\prod_{t\in J_{0,r-2}} (\chi(\alpha_1^{1\rhd}, \lambda)q_1^{-2t} - 1))(E_1^\vee)^{r-1} L_{\overline{(r-1)\alpha_1^{1\rhd}-\lambda}} E_1^{r-1},$$

and

$$(9.4)\quad \begin{aligned} Y := {} & \sqrt{\chi}(\alpha_1, -(s_1(\lambda) + 2(r-1)\alpha_1))(-1)^{r-1} \\ & \cdot(\textstyle\prod_{t\in J_{0,r-2}}(\chi(\alpha_1, s_1(\lambda) + 2(r-1)\alpha_1)q_1^{-2t} - 1)) \\ & \cdot(E_1^\vee)^{r-1} L_{\overline{(r-1)\alpha_1-(s_1(\lambda)+2(r-1)\alpha_1)}} E_1^{r-1} \\ = {} & \sqrt{\chi}(\alpha_1, s_1(\lambda))^{-1} q_1^2 (-1)^{r-1} \\ & \cdot(\textstyle\prod_{t\in J_{0,r-2}}(\chi(\alpha_1, s_1(\lambda))q_1^{4(r-1)-2t} - 1)) \\ & \cdot(E_1^\vee)^{r-1} L_{\overline{(r-1)\alpha_1-(s_1(\lambda)+2(r-1)\alpha_1)}} E_1^{r-1} \\ = {} & \sqrt{\chi}(\alpha_1, s_1(\lambda))^{-1} q_1^2 (-1)^{r-1} \\ & \cdot q_1^{4(r-1)^2-(r-2)(r-1)}(-1)^{r-1}\chi(\alpha_1, s_1(\lambda))^{r-1} \\ & \cdot(\textstyle\prod_{t\in J_{0,r-2}}(\chi(\alpha_1, s_1(\lambda))q_1^{2(t+2)} - 1)) \\ & \cdot(E_1^\vee)^{r-1} L_{\overline{(r-1)\alpha_1-(s_1(\lambda)+2(r-1)\alpha_1)}} E_1^{r-1} \\ = {} & (-1)^{r-1} q_1^{4(r-1)^2-r(r-3)} \chi(\alpha_1, s_1(\lambda))^{r-2} \\ & \cdot\sqrt{\chi}(\alpha_1, s_1(\lambda))(-1)^{r-1}(\textstyle\prod_{t\in J_{0,r-2}}(\chi(\alpha_1, s_1(\lambda))q_1^{-2t} - 1)) \\ & \cdot(E_1^\vee)^{r-1} L_{\overline{(r-1)\alpha_1-(s_1(\lambda)+2(r-1)\alpha_1)}} E_1^{r-1}. \end{aligned}$$

Then we have

$$
\begin{aligned}
&T_1(Y) = \\
&T_1(\sqrt{\chi}(\alpha_1^{1\triangleright},-\lambda)(-1)^{r-1}(\prod_{t\in J_{0,r-2}}(\chi(\alpha_1^{1\triangleright},\lambda)q_1^{-2t}-1))(E_1^{\vee})^{r-1}L_{\overline{(r-1)\alpha_1^{1\triangleright}-\lambda}}E_1^{r-1}) \\
&\quad = \sqrt{\chi}(-\alpha_1,-s_1(\lambda))(-1)^{r-1}(\prod_{t\in J_{0,r-2}}(\chi(-\alpha_1,s_1(\lambda))q_1^{-2t}-1)) \\
&\qquad\qquad \cdot(L_{-\overline{\alpha_1}}E_1)^{r-1}L_{\overline{-(r-1)\alpha_1-s_1(\lambda)}}(E_1^{\vee}L_{\overline{\alpha_1}})^{r-1} \\
&\quad \equiv \sqrt{\chi}(\alpha_1,s_1(\lambda))(-1)^{r-1}(\prod_{t\in J_{0,r-2}}(\chi(-\alpha_1,s_1(\lambda))q_1^{-2t}-1)) \\
&\qquad\qquad \cdot\chi(\alpha_1,(r-1)\alpha_1+s_1(\lambda))^{2(r-1)} \\
&\qquad\qquad \cdot(E_1^{\vee})^{r-1}L_{\overline{(r-1)\alpha_1-(s_1(\lambda)+2(r-1)\alpha_1)}}E_1^{r-1} \quad (\mathrm{mod}\ \oplus_{k\in J_{0,r-2}}(E_1^{\vee})^kU^0E_1^k\ ) \\
&\quad = (-1)^{r-1}q_1^{r(r-3)}\chi(\alpha_1,s_1(\lambda))^rY \\
&\quad = \chi(\alpha_1,s_1(\lambda))^rY.
\end{aligned}
$$

Assume that $\{r-1\}_{q_1^2}!\neq 0$ and $\chi(\alpha_1^{1\triangleright},\lambda)=q_1^{2(r-1)}$. Note that $s_1(\lambda)=\lambda$, since in this stage, s_1 is the identity map. Let

$$
X^{1\triangleright} := \sqrt{\chi}(\alpha_1^{1\triangleright},-\lambda)(-1)^{r-1}(\prod_{t\in J_{1,r-1}}(q_1^{2t}-1))(E_1^{\vee})^{r-1}L_{\overline{(r-1)\alpha_1^{1\triangleright}-\lambda}}E_1^{r-1}, \tag{9.5}
$$

and

$$
\begin{aligned}
X &:= \sqrt{\chi}(\alpha_1,-(s_1(\lambda)+2(r-1)\alpha_1))(-1)^{r-1}(\textstyle\prod_{t\in J_{1,r-1}}(q_1^{2t}-1)) \\
&\quad\cdot(E_1^{\vee})^{r-1}L_{\overline{(r-1)\alpha_1-(s_1(\lambda)+2(r-1)\alpha_1)}}E_1^{r-1} \\
&= \sqrt{\chi}(\alpha_1^{1\triangleright},-\lambda)(-1)^{r-1}(\textstyle\prod_{t\in J_{1,r-1}}(q_1^{2t}-1))(E_1^{\vee})^{r-1}L_{\overline{-(r-1)\alpha_1-\lambda}}E_1^{r-1}.
\end{aligned} \tag{9.6}
$$

Then by the same argument as above, we have

$$
T_1(X^{1\triangleright}) = X \quad (\mathrm{mod}\ \oplus_{k\in J_{0,r-2}}(E_1^{\vee})^kU^0E_1^k\ ). \tag{9.7}
$$

Then the statement of this theorem follows by an argument similar to that for Proof of Theorem 8. □

10 $\mathfrak{osp}(3|2)$

Assume that $N=3$ and $\ell=2$. Let $p_1:=0$, and $p_2:=1$. Define the symmetric bi-additive map $((\,,\,)):\mathbb{Z}\Pi'\times\mathbb{Z}\Pi'\to\mathbb{Z}$ by $((\epsilon_i,\epsilon_j)):=(1-\delta_{i,3})(1-\delta_{j,3})(-1)^{\delta_{ij}p_i}$. Assume that $\sqrt{\chi}(\epsilon_3,\epsilon_3)=\sqrt{-1}$, $\sqrt{\chi}(\epsilon_3,\epsilon_r)=\sqrt{\chi}(\epsilon_r,\epsilon_3)=1$ hold for all $r\in J_{1,2}$, and $\sqrt{\chi}(\epsilon_i,\epsilon_j)=q^{\frac{((\epsilon_i,\epsilon_j))}{2}}$ for all $i,\,j\in J_{1,3}$. Then there exists an additive group isomorphism $\mathbb{Z}^2\times(\mathbb{Z}/4\mathbb{Z})\to\overline{\mathbb{Z}\Pi'}$, $(m_1,m_2,m_3+4\mathbb{Z})\mapsto\sum_{t\in J_{1,3}}m_t\bar{\epsilon}_t$, where

$m_t \in \mathbb{Z}$. Assume that $\alpha_1 = \epsilon_1 - \epsilon_2 + \epsilon_3$ and $\alpha_2 = \epsilon_2$. Assume that $\tilde{\rho} = -\epsilon_1 + \epsilon_2 + \epsilon_3$ We also denote this U by $U_q(\mathfrak{osp}(3|2))$.

We have $\alpha_1^{1\triangleright} = -\alpha_1 = -\epsilon_1 + \epsilon_2 - \epsilon_3$, and $\alpha_2^{1\triangleright} = \alpha_1 + \alpha_2 = \epsilon_1 + \epsilon_3$. Let $\tilde{\rho}^{1\triangleright} := \epsilon_1 - \epsilon_2 - \epsilon_3$. Then $\chi(\tilde{\rho}^{1\triangleright}, \alpha_i^{1\triangleright}) = \chi(\alpha_i^{1\triangleright}, \alpha_i^{1\triangleright})$.

Let $z \in \mathfrak{z}(U)$. Assume that $\Phi(z) = \sum_{m_1,m_2 \in \mathbb{Z}, m_3 \in J_{0,3}} x_{m_1,m_2,m_3} L_{m_1\overline{\epsilon}_1 + m_2\overline{\epsilon}_2 + m_3\overline{\epsilon}_3}$ with $x_{m_1,m_2,m_3} \in \mathbb{C}$. By Theorem 8 (1), we have $q^{\frac{m_2}{2}} x_{m_1,-m_2,m_3} = q^{\frac{-m_2}{2}} x_{m_1,m_2,m_3}$. By Theorem 8 (2), we conclude that if $m_1 + m_2 \neq 0$ or $m_3 \in \{1,3\}$, then $\sum_{k\in\mathbb{Z}} x_{m_1+2k,m_2-2k,m_3+2k} = 0$. By Theorem 12, we have $m_3 - m_1 \in 2\mathbb{Z}$, and we have $x_{-m_1,m_2,-m_1+2m_3} = q^{m_1} x_{m_1,m_2,m_1+2m_3}$. Let $n_1 \in \mathbb{N}+1$, $n_2 \in \mathbb{Z}_{\geq 0}$ and $n_3 \in J_{0,1}$. Let $\lambda := n_1\epsilon_1 + n_2\epsilon_2 + (n_1 + 2n_3)\epsilon_3$. Then $\chi(\alpha_1, \lambda) = q^{n_1+n_2}(-1)^{n_1} \neq 1$, $\chi(\alpha_2, \lambda) = q^{-n_2} = \chi(\alpha_2, \alpha_2)^{n_2}$, and $\chi(\alpha_2^{1\triangleright}, \lambda - \alpha_1) = q^{n_1-1}(-1)^{n_1-1} = \chi(\alpha_2^{1\triangleright}, \alpha_2^{1\triangleright})^{n_1-1}$. Recall $T_1 : U^{1\triangleright} \to U$. Recall $v_{\overline{\lambda}} \in V(\lambda)$. Then $E_1^{\vee} v_{\overline{\lambda}} \neq 0$, and $U_\mu^+ v_{\overline{\lambda}} = T_1((U^{1\triangleright})^+_{s_i(\mu)+\alpha_1^{1\triangleright}}) E_1^{\vee} v_{\overline{\lambda}}$ hold for all $\mu \in \mathbb{Z}\Pi$ (in this stage, s_i means the identity map). Then in $V(\lambda)$, we have $(E_2^{\vee})^k v_{\overline{\lambda}} \neq 0$ ($k \in J_{0,n_2}$), $(E_2^{\vee})^{n_2+1} v_{\overline{\lambda}} = 0$, and $T_1(E_2^{\vee})^r E_1^{\vee} v_{\overline{\lambda}} \neq 0$ ($r \in J_{0,n_1-1}$), $T_1(E_2^{\vee})^{n_1} E_1^{\vee} v_{\overline{\lambda}} = 0$. Let $\alpha \in \mathbb{Z}_{\geq 0}\Pi$ be such that $U^-_{-\alpha} v_\lambda \neq \{0\}$ holds in $V(\lambda)$. Then $\alpha = x_1\alpha_1 + x_2\alpha_2 = \alpha_1 + y_1\alpha_1^{1\triangleright} + y_2\alpha_2^{1\triangleright} = (1 - y_1 + y_2)\alpha_1 + y_2\alpha_2$ for some x_1, x_2, y_1, $y_2 \in \mathbb{Z}_{\geq 0}$. Then $x_2 = y_2 = x_1 + y_1 - 1$. Then $x_2 \geq x_1 - 1 + \delta_{x_1,0}$. Note $\alpha = x_1\epsilon_1 + (x_2 - x_1)\epsilon_2 + x_1\epsilon_3$. By (4.4), we conclude

$$
\begin{aligned}
&\Phi(\zeta^{-1}(f_{V(\lambda)})) \\
&\in \sum_{k_1 \in J_{0,n_1-2}} (\mathbb{C}^\times L_{(n_1-2-2k_1)\overline{\epsilon_1} + (n_2+2)\overline{\epsilon}_2 + (n_1+2n_3)\overline{\epsilon_3}} \\
&\qquad + \mathbb{C}^\times L_{(n_1+2-2k_1)\overline{\epsilon_1} + (-n_2-2)\overline{\epsilon_2} + (n_1+2n_3)\overline{\epsilon_3}}) \\
&\quad \sum_{k_2 \in J_{0,n_2}, k_3 \in J_{0,n_1}} \mathbb{C}^\times L_{(n_1-2k_3)\overline{\epsilon_1} + (n_2-2k_2)\overline{\epsilon_2} + (n_1+2n_3-2k_3)\overline{\epsilon_3}}.
\end{aligned}
$$

Finally we have

Theorem 13. *Let U be as above. Then we have*

$$
\begin{aligned}
&\mathfrak{z}(U_q(\mathfrak{osp}(3|2))) \\
&= \bigoplus_{y \in J_{0,1}} \mathbb{C}L_{2y\overline{\epsilon_3}} \oplus \bigoplus_{n_1 \in \mathbb{N}+1, n_2 \in \mathbb{Z}_{\geq 0}, n_3 \in J_{0,1}} \mathbb{C}\zeta^{-1}(f_{V(n_1\overline{\epsilon_1} + n_2\overline{\epsilon_2} + (n_1+2n_3)\overline{\epsilon_3})}).
\end{aligned}
$$

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